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TOPOLOGICAL OBSTRUCTIONS TO CERTAIN LIE GROUP ACTIONS ON MANIFOLDS

PISHENG DING

ABSTRACT. Given a smooth closed S^1-manifold M, this article studies the extent to which certain numbers of the form $(f^*(x) \cdot P \cdot C)[M]$ are determined by the fixed-point set M^{S^1}, where $f : M \to K(\pi_1(M), 1)$ classifies the universal cover of M, $x \in H^*(\pi_1(M); \mathbb{Q})$, P is a polynomial in the Pontrjagin classes of M, and C is in the subalgebra of $H^*(M; \mathbb{Q})$ generated by $H^2(M; \mathbb{Q})$. When $M^{S^1} = \varnothing$, various vanishing theorems follow, giving obstructions to certain fixed-point-free actions. For example, if a fixed-point-free S^1-action extends to an action by some semisimple compact Lie group G, then $(f^*(x) \cdot P \cdot C)[M] = 0$. Similar vanishing results are obtained for spin manifolds admitting certain S^1-actions.

1. INTRODUCTION

Terminological convention. Throughout this article, unless indicated otherwise, "*Lie group*" and "*G*" signify "*positive-dimensional connected compact Lie group*", "*manifold*" and "*M*" signify "*oriented closed smooth manifold*", and, "*action*" signifies "*smooth action*".

Given a manifold M, let π denote $\pi_1(M)$ and let $f : M \to K(\pi, 1)$ classify the universal covering space of M; let C be an element of the subalgebra of $H^*(M; \mathbb{Q})$ generated by $H^2(M; \mathbb{Q})$, and let P be a rational polynomial in the Pontrjagin classes of M.

Suppose M is an S^1-manifold. Let $o : S^1 \to M$ be the orbit of some basepoint in M. As a standard fact, $o_*(\pi_1(S^1))$ is a central subgroup of π. Let $\pi' = \pi / o_*(\pi_1(S^1))$, and let $\rho : \pi \to \pi'$ be the quotient map. Let $x \in H^*(\pi; \mathbb{Q})$, $y \in H^*(\pi'; \mathbb{Q})$.

This article mainly studies the extent to which certain numbers of the form $(f^*(x) \cdot P \cdot C)[M]$ are determined by the fixed-point set M^{S^1}. When $M^{S^1} = \varnothing$, various vanishing theorems follow, giving obstructions to certain fixed-point-free actions. My investigation draws upon the Atiyah-Singer's index theory (mainly the G-signature theorem), some equivariant bundle theory of [8], and [3]'s work relating an effective G-action on M to $\pi_1(M)$.

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In this Introduction, a number of salient results of this work are described along with a review of some related existing results. I first give an overview of several vanishing statements, which form various parts of Theorem A. The proof of parts of Theorem A relies in an essential way on some localization results stated in Theorem C, the exposition of which forms a major portion of this article. I mention in Theorem D two vanishing propositions about certain spin S^1-manifolds of a similar flavor. Theorem B provides a digression related to Theorem A($3'$).

Using the then newly invented index theory, [2] obtains "residue formulae" for the Pontrjagin numbers of M in terms of the submanifold $M^{S^1} \hookrightarrow M$ and the S^1-action on its normal bundle; this in particular implies that, when $M^{S^1} = \varnothing$, all the Pontrjagin numbers of M vanish. Combining this result and [3]'s work, one immediately obtains

Theorem A(1). *Suppose $M^{S^1} = \varnothing$. Then, $(f^* \rho^* (y) \cdot P)[M] = 0$; if, in addition, $\operatorname{Im}\left(\pi_1 (S^1) \xrightarrow{o_*} \pi_1 (M)\right)$ is finite, then $(f^* (x) \cdot P)[M] = 0$.*

Suppose $M^{S^1} = \varnothing$. Using differential-geometric method, [12] shows that, if $H^1 (M) = \{0\}$, then $C[M] = 0$; using index theory and some equivariant bundle theory of [8], [5] shows that, if $\pi_1 (M)$ is finite, then $\left(c^k \cdot \mathbf{L}(M)\right)[M] = 0$ where $c \in H^2 (M; \mathbb{Q})$ and $\mathbf{L}(M)$ is the Hirzebruch L-class of M. Generalizing these two propositions is

Theorem A(2). *Suppose $M^{S^1} = \varnothing$. If $H^1 (M) = \{0\}$, then $(f^* \rho^* (y) \cdot P \cdot C)[M] = 0$; if, in addition, $\operatorname{Im}\left(\pi_1 (S^1) \xrightarrow{o_*} \pi_1 (M)\right)$ is finite, then $(f^* (x) \cdot P \cdot C)[M] = 0$.*

We show in the next result that, under some hypothesis different from that in Theorem A(2), the same conclusion can be drawn.

Theorem A(3). *Suppose $M^{S^1} = \varnothing$. If the S^1-action on M extends to an action by a semisimple Lie group G, then $(f^* (x) \cdot P \cdot C)[M] = 0$.*

Thus, the nonvanishing of any number of the form $(f^* (x) \cdot P \cdot C)[M]$ is an obstruction to extending any fixed-point-free S^1-action to an action by a semisimple Lie group G. Said differently, this yields

Theorem A($3'$). *If a semisimple Lie group G acts on M with some $g \in G$ acting freely, then $(f^* (x) \cdot P \cdot C)[M] = 0$.*

Note that, if $f^* (x)[M^m] = 0$ for all $x \in H^* (\pi; \mathbb{Q})$, then $f_* [M^m] = 0 \in H_m (\pi; \mathbb{Q})$. Hence, Theorem A($3'$) implies that a nonzero $f_* [M]$ is an obstruction to certain G-actions with G semisimple. In fact, a stronger statement holds, showing that a nonzero $f_* [M]$ is an obstruction to *any* G-action with G semisimple. I mention two earlier results along this direction.

[4] shows that, if M is aspherical, then the only Lie groups acting effectively on M are tori with dimensions bounded above by rank $\mathcal{Z}(\pi)$ ("$\mathcal{Z}$" signifying "the center of"), and such effective torus-actions must be fixed-point-free and have only finite isotropy subgroups. [12] shows that, if there exist $\omega_1, \cdots, \omega_m \in H^1 (M)$ such that $\prod_j \omega_j [M^m] \neq 0$, then the only Lie groups acting effectively on M are tori. Note that weaker than the hypotheses of both of these two statements is the condition that $f_* [M] \neq 0$; thus the following theorem generalizes them.

Theorem B. *Suppose M^m satisfies the condition that $f_* [M] \neq 0 \in H_m(\pi; \mathbb{Q})$.*

(1) *If G is semisimple, then any G-action on M is trivial.*
(2) *If G acts effectively on M, then G is a torus with $\dim G \leq \operatorname{rank} \mathcal{Z}(\pi)$ and all isotropy subgroups are finite.*
(3) *If G acts nontrivially on M, then $M^S = \varnothing$ for some circle subgroup $S \subset G$; hence, Theorems* A(1) *and* A(2) *apply.*

The core of the proof of Theorems A(2) and A(3) is some "localization analysis" which the next theorem deals with. Given an S^1-manifold M, Theorem C treats "localization" of $(P \cdot C)[M]$, i.e., determination of such numbers in terms of fixed-point data. Let $R = \left(c^k \cdot P\right)[M]$ where $c \in \operatorname{Im}\left(H^2(M) \to H^2(M; \mathbb{Q})\right)$. It turns out, as a matter of algebra, that to compute $(P \cdot C)[M]$, it suffices to only compute R. Therefore, the next theorem is only stated for R.

Let $\mathcal{L}(c)$ be the complex line bundle over M whose first Chern class is c. Call the linear S^1-action on a fiber of the normal bundle over M^{S^1} an isotropy S^1-representation.

Theorem C. *Let M be a S^1-manifold.*

(1) *If $H^1(M) = 0$, or, if the S^1-action extends to a G-action with $\pi_1(G) = \{0\}$, then R can be determined in terms of: the submanifold $M^{S^1} \hookrightarrow M$, the isotropy S^1-representations, and the restriction to $\mathcal{L}(c)|_{M^{S^1}}$ of any lifted S^1-action on $\mathcal{L}(c)$.*
(2) *If, for some semisimple Lie group G, the S^1-action extends to a G-action with each component of M^{S^1} containing a point of M^G, then R can be determined in terms of the submanifold $M^{S^1} \hookrightarrow M$ and the isotropy S^1-representations.*
(3) *Let F_0 be the union of those components of M^{S^1} with codimensions congruent to* $0 \bmod 4$. *Then, under the same condition as in* (2), $(\mathbf{L}(M) \cdot C)[M]$ *can be determined in terms of the submanifold $F_0 \hookrightarrow M$ (and is independent of the isotropy S^1-representations). In particular, if, in addition to the stated condition, $F_0 = \varnothing$, then $(\mathbf{L}(M) \cdot C)[M] = 0$.*

(The condition on the extension of the S^1-action to some G-action appearing in Theorem C is discussed in the Remark below.)

Suppose M is a spin manifold admitting a nontrivial S^1-action. [1] shows that its $\hat{\mathbf{A}}$-genus vanishes; [3] proves that certain higher $\hat{\mathbf{A}}$-genera vanish. (A higher $\hat{\mathbf{A}}$-genus is a number of the form $\left(f^*(x) \cdot \hat{\mathbf{A}}(M)\right)[M]$.) I show

Theorem D. *Let M be a spin manifold admitting a nontrivial S^1-action which extends to a G-action with G a semisimple Lie group.*

(1) *If each component of M^{S^1} contains a point of M^G, then $\left(\hat{\mathbf{A}}(M) \cdot C\right)[M] = 0$.*
(2) *If $M^{S^1} = M^G$, then $\left(f^*(x) \cdot \hat{\mathbf{A}}(M) \cdot C\right)[M] = 0$.*

Remark. The condition that *for some semisimple G the S^1-action extends to a G-action with each component of M^{S^1} containing a point of M^G* is essential for the validity of parts of Theorems A, C and D. For example, realizing S^2 as the homogeneous space $SU(2)/S$ where $S \subset SU(2)$ is a maximal torus (which is a circle subgroup), one has a fixed-point-free $SU(2)$-action on S^2 but $\left(S^2\right)^S$ is nonempty; now there indeed are nonvanishing cohomology classes in $H^2\left(S^2\right)$, which invalidates the conclusions of Theorems A($3'$), C(3) and D. More generally, given a semisimple Lie group G and a closed proper subgroup H containing a maximal torus T of G, G/H is an even-dimensional manifold with a fixed-point-free G-action but $(G/H)^T$ is certainly nonempty; often then, the characteristic numbers appearing in Theorems A($3'$), C(3) and D are nonzero.

Now we discuss how this condition might be realized. If M admits a semifree $SU(2)$-action, then, for any circle subgroup $S \subset SU(2)$, $M^S = M^{SU(2)}$; hence the condition is met. ($SU(2)$ happens to be the only semisimple Lie group that can act semifreely on manifolds.)

In general, given a G-action, one may ask when there will be a circle subgroup $S \subset G$ with $M^S = M^G$.

For T a maximal torus in G, there is a circle subgroup $S \subset T$ with $M^S = M^T$; thus, any condition relating M^S and M^G can be reformulated as a condition relating M^T and M^G. It is shown in [6] that an even-codimensional component F of M^G is also a component of M^T *if and only if* the equivariant Euler class of the normal bundle of $F \hookrightarrow M$ satisfies some "invertibility" condition, which I now describe.

Let F be a $2k$-codimensional component of M^G and $N(F) \to F$ the normal bundle of F in M. The equivariant Euler class of $N(F)$, denoted by $e_G(N(F))$, is an element of

$$H^{2k}(EG \times_G F; \mathbb{Q}) = H^{2k}(BG \times F; \mathbb{Q}) \simeq \bigoplus_{j=0}^{k} H^{2k-2j}(BG; \mathbb{Q}) \otimes H^{2j}(F; \mathbb{Q}).$$

So we may write

$$e_G(N(F)) = \alpha_{2k}(F) \otimes 1 + \sum \alpha \otimes \beta,$$

where $\alpha_{2k}(F) \in H^{2k}(BG; \mathbb{Q})$, $\alpha \in H^{2(k-j)}(BG; \mathbb{Q})$ and $\beta \in H^{2j}(F; \mathbb{Q})$ with $j > 0$. [6] shows that F is also a connected component of M^T *if and only if* $\alpha_{2k}(F) \neq 0$. Hence, if every component of M^G satisfies this condition and if M^T has no more components than M^G, then $M^G = M^T$.

2. Preliminaries

The purpose of this section is three-fold: to review some existing results, to deduce some immediate consequences of them relevant to our developments, and in doing so, to establish notation and terminology.

2.1. **Equivariant bundle theory.**

We review the problem of lifting a group action on X to an action on the total space of a bundle over X. We will be primarily interested in the case of an S^1-bundle (or equivalently, a complex line bundle).

2.1.1. *Lifting G-actions in fiber bundles.* Let X be a G-space and $E \overset{\mathrm{Pr}}{\to} X$ a principal H-bundle, G and H being Lie groups (not necessarily connected). Some natural questions arising from this situation are whether, when, and how the G-action on X can be lifted to a G-action on E commuting with the H-action on E. (The commutativity of the G-action and H-action on E is equivalent to the condition that G acts on E by (H-bundle)-isomorphisms, which is to be assumed whenever we speak of a lifted G-action on the total space of a principal H-bundle.) For example, when G acts isometrically on a Riemannian manifold M^m, the principal $O(m)$-bundle P associated with the Riemannian structure of M^m always admits a natural lifting, the G-action on P being induced by that on M.

For a G-space X having the homotopy type of a CW-complex, let $\mathfrak{B}_{(G,H)}(X)$ be the set of equivalence classes of G-(H-bundle)s over X, a G-(H-bundle) over X being a principal H-bundle whose total space admits a lifted G-action, and, two G-(H-bundle)s being equivalent if there is a G-equivariant (H-bundle)-equivalence between them.

For a space Y having the homotopy type of a CW-complex, let $\mathfrak{B}_H(Y)$ be the set of equivalence classes of principal H-bundles over Y.

Given a G-space X, there is an obvious map

$$\phi : \mathfrak{B}_{(G,H)}(X) \longrightarrow \mathfrak{B}_H(X)$$

which forgets the G-equivariant structure, yielding the underlining H-bundle of a G-(H-bundle). $\operatorname{Im}\phi$ is precisely the set of equivalence classes of H-bundles which admit liftings of the G-action on X. Let $X_G = EG \times_G X$. Choosing a basepoint in EG gives rise to an inclusion $i : X \to X_G$, which induces a restriction map:

$$i^* : \mathfrak{B}_H(X_G) \longrightarrow \mathfrak{B}_H(X).$$

Given a G-(H-bundle) $E \overset{\mathrm{Pr}}{\to} X$, the Borel construction gives $E_G \overset{\mathrm{Pr}_G}{\longrightarrow} X_G$, an H-bundle over X_G. This produces a map

$$\psi : \mathfrak{B}_{(G,H)}(X) \longrightarrow \mathfrak{B}_H(X_G).$$

Evidently, $i^* \circ \psi = \phi$.

The following proposition is shown in [8]:

Proposition 2.1 ([8])**.** *When H is abelian, ψ is a bijection and hence* $\operatorname{Im} i^* = \operatorname{Im}\phi$.

2.1.2. *Lifting connected group actions in S^1-bundles.* We now specialize the result in Proposition 2.1 to the case in which $H = S^1$ and G is connected. Identifying an S^1-bundle with its first Chern class, we have

$$\mathfrak{B}_{S^1}(Y) \simeq \left[Y, BS^1\right] \simeq H^2(Y).$$

By Proposition 2.1, $\operatorname{Im} i^* = \operatorname{Im}\phi$. We may regard i^* as the restriction

$$i^* : H^2(X_G) \longrightarrow H^2(X),$$

the image of which can readily be studied by the Leray-Serre spectral sequence associated with the fiber bundle $X \overset{i}{\hookrightarrow} X_G \overset{p}{\to} BG$. As $\pi_1(BG) \simeq \pi_0(G) = \{0\}$, this bundle has a simple system of local coefficients on the base BG.

Corollary 2.2. *Let G be a Lie group acting on a space X. Suppose $H^3(BG) = \{0\}$ and $H^1(X) = \{0\}$. Let $E \to X$ be an S^1-bundle. Then $E \to X$ can be given a G-equivariant structure, i.e., the G-action on X can be lifted to a G-action on E. Furthermore, the set of equivalence classes of the G-(S^1-bundle)s resulting from all possible liftings is in bijective correspondence with $H^2(BG)$.*

Proof. In the fiber bundle $X \overset{i}{\hookrightarrow} X_G \overset{p}{\to} BG$, $H^1(BG) = \{0\}$ by simple connectivity of BG (which is equivalent to the connectivity of G), and, $H^1(X) = \{0\}$ by hypothesis; thus we have the following exact sequence, which is a portion of the Serre exact sequence derived from the Serre spectral sequence:

$$\{0\} = H^1(X) \longrightarrow H^2(BG) \overset{p^*}{\longrightarrow} H^2(X_G) \overset{i^*}{\longrightarrow} H^2(X) \longrightarrow H^3(BG) = \{0\};$$

see [10]. By the exactness of this sequence, i^* is surjective; hence so is ϕ, proving the first statement. Since $i^* \circ \psi = \phi$ and ψ is a bijection, the multitude of liftings (i.e., the multitude of $\phi^{-1}[E]$) is the same as that of $\ker i^* = \operatorname{Im} p^* \simeq H^2(BG)$, as claimed. □

Corollary 2.3. *Let G be a simply connected Lie group acting on a space X. Then every S^1-bundle over X admits a unique G-equivariant structure, i.e., $\phi : \mathfrak{B}_{(G,S^1)}(X) \to \mathfrak{B}_{S^1}(X)$ is an isomorphism.*

Proof. If G is 1-connected, G is then also 2-connected as $\pi_2(G) = \{0\}$ for any compact Lie group. Thus, for $1 \le j \le 3$, $\pi_j(BG) \simeq \pi_{j-1}(G) = \{0\}$, and by the Hurewicz theorem, $H^j(BG) = \{0\}$. Under these conditions, we have the following exact sequence, which is a portion of the Serre exact sequence:

$$\{0\} = H^2(BG) \xrightarrow{p^*} H^2(X_G) \xrightarrow{i^*} H^2(X) \longrightarrow H^3(BG) = \{0\},$$

implying that $i^* : H^2(X_G) \to H^2(X)$ is an isomorphism. As $i^* \circ \psi = \phi$, $\phi : \mathfrak{B}_{(G,S^1)}(X) \to \mathfrak{B}_{S^1}(X)$ is an isomorphism, as claimed. □

Next, we make an observation regarding E^G, the fixed-point set of the total space of a G-(S^1-bundle). As above, let X be a G-space and $\Pr : E \to X$ be an S^1-principal bundle. As we see in the above discussion, liftings, if they exist, may not be unique. In general, the lifted G-action on the fiber $\Pr^{-1}[a]$ over a fixed point $a \in X^G$ depends on the global lifting and may not always be the trivial one. However, when G is semisimple, the next proposition shows that any lifted G-action on $\Pr^{-1}[a]$ for $a \in X^G$ is always the trivial action.

Proposition 2.4. *Let G be a semisimple Lie group acting on a space X and let $\Pr : E \to X$ be a G-(S^1-bundle). Then, $E^G = \Pr^{-1}[X^G]$.*

Proof. Over each connected component X^G_ν of X^G, the G-action on $\Pr^{-1}[X^G_\nu]$ produces a 1-dimensional complex representation $\rho_\nu : G \to S^1$. It suffices to show that there is no nontrivial Lie group homomorphism $\rho : G \to S^1$, i.e., that G has no connected abelian quotient, which follows immediately from the Lie-algebra-theoretic definition of semisimplicity. □

Remark 2.5. A principal S^1-bundle E over X is the "same" as a complex line bundle $\mathcal{L} \to X$, and, a lifted G-action on $\mathcal{L}$ is simply a homomorphism $G \to \mathrm{Iso}(\mathcal{L})$ with a prescribed G-action on the zero section of $\mathcal{L}$. The results in this section can then be restated in terms of complex line bundles.

2.2. **The G-signature and G-spin theorems.** We review those parts of the index theory that are related to the present study. We first briefly describe the construction of the twisted G-signature of a G-manifold, then state the Atiyah-Singer's G-signature formula, and finally include a description of the G-spinor-index.

2.2.1. *The G-signature.* Let M^{2n} be a Riemannian manifold and G be a Lie group acting on M by orientation-preserving isometries. Let $\Lambda^*(T^* \otimes \mathbb{C})$ be the bundle of complex exterior algebras of the complexified cotangent bundle. There is a bundle homomorphism $\tau : \Lambda^k(T^* \otimes \mathbb{C}) \to \Lambda^{2n-k}(T^* \otimes \mathbb{C})$ defined via the Hodge star $*$:

$$\tau(a) = i^{k(k-1)+n} * (a) \quad \text{for} \quad a \in \Lambda^k(T^* \otimes \mathbb{C}).$$

Note that τ^2 is the identity. For $E \to M$ a G-equivariant complex vector bundle, let $\Omega^*_E(M)$ be the space of complex differential forms with coefficients in E, i.e., the space of smooth sections of $\Lambda^*(T^* \otimes \mathbb{C}) \otimes E$. τ canonically induces an operator τ_E on $\Omega_E^*(M)$. τ_E is an involution. Let $\Omega_E^{\pm}(M)$ be the ± 1-eigenspaces of τ_E. Let

$$D = d + d^* : \Omega^*(M) \longrightarrow \Omega^*(M),$$

where d is the de Rham operator and d^* its formal adjoint (with respect to the inner product on Ω^* induced by the Riemannian metric). Upon equipping E with a connection, D induces an operator D_E on $\Omega_E^*(M)$ and D_E anticommutes with τ_E. Let $D_E^+ : \Omega_E^+(M) \to \Omega_E^-(M)$ be the restriction of D_E to $\Omega_E^+(M)$. Then the twisted G-signature, $\mathrm{Sign}_G(M;E)$, is defined to be the G-index of D_E^+ :

$$\mathrm{ind}_G(D_E^+) = [\ker D_E^+] - [\mathrm{coker}\, D_E^+] \in R(G)\,.$$

This index invariant of D_E^+ is independent of the choice of connections on E, which justifies our notational suppression of the chosen connection. Regarding elements of $R(G)$ as character functions $G \to \mathbb{C}$, we denote the value of $\mathrm{Sign}_G(M;E)$ at $g \in G$ by $\mathrm{Sign}_G(M;E)(g)$.

Index theory gives a localization formula for $\mathrm{Sign}_G(M;E)(g)$, i.e., a formula in terms of the topology of M^g, the g-action on the normal bundle N of M^g, the bundle $E|_{M^g}$ and the g-action on $E|_{M^g}$. We now describe this formula in the special case where $g \in G$ is a topological generator of a toral subgroup $T \subset G$.

As g generates T, $M^g = M^T$. Hence $\mathrm{Sign}_G(M;E)(g)$ depends only on the T-action, i.e.,

$$\mathrm{Sign}_G(M;E)(g) = \mathrm{Sign}_T(M;E)(g).$$

E enters the formula via a class $\mathrm{ch}(E|_{M^g})(g) \in H^*(M^g;\mathbb{C})$, which we now describe. As M^g is a trivial T-space, we can identify $K_T(M^g)$ with $K(M^g) \otimes R(T)$ and define $\mathrm{ch}(\ )(g) : K_T(M^g) \to H^*(M^g;\mathbb{C})$ as the map

$$\mathrm{ch} \otimes \chi(g) : K(M^g) \otimes R(T) \longrightarrow H^*(M^g;\mathbb{C}) \otimes \mathbb{C} \simeq H^*(M^g;\mathbb{C})\,.$$

Over each $x \in M^g$, the fiber N_x of N is a real T-module, which can be decomposed into a sum of two-dimensional irreducible subrepresentations, each of which is of the following type and hence has a canonical complex structure:

$$g \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \text{or} \quad g \mapsto e^{i\theta}, \quad \theta \in (0,\pi)\,.$$

For $\theta \in (0,\pi)$, let $N_x(\theta)$ be the sum of those irreducible subrepresentations given by $g \mapsto e^{i\theta}$. Then,

$$N_x = \bigoplus_{\theta \in (0,\pi)} N_x(\theta),$$

where each $N_x(\theta)$ has a canonical complex structure. (There are only finitely many θ for which $N_x(\theta)$ is nontrivial.) Let $M^g = \bigcup_\nu M_\nu^g$, where M_ν^g are the connected components of M^g, and let N_ν denote the normal bundle of M_ν^g. The fiberwise decomposition leads to a bundle decomposition

$$N_\nu = \bigoplus_{\theta \in (0,\pi)} N_\nu(\theta).$$

The canonical orientation of N_ν induced by the complex structure gives an orientation of M_ν^g such that $T(M_\nu^g) \oplus N_\nu = (TM)|_{M_\nu^g}$.

For a manifold X^m, the Hirzebruch $\hat{\mathbf{L}}$-class, denoted by $\hat{\mathbf{L}}(X)$, is defined by

$$\hat{\mathbf{L}}(X) = \prod_{j=1}^{\left[\frac{m+1}{2}\right]} \frac{x_j/2}{\tanh(x_j/2)}$$

where $\left\{+x_j, -x_j | 1 \le j \le \left[\frac{m+1}{2}\right]\right\}$ are the formal Chern roots of either $TX^m \otimes \mathbb{C}$ or $(TX^m \oplus \mathbf{1}_{\mathbb{R}}) \otimes \mathbb{C}$ according to whether m is even or odd respectively. We write $\hat{L}_k(X)$ for the $4k$-dimensional component of $\hat{\mathbf{L}}(X)$.

We are now ready to write the Atiyah-Singer's G-signature formula for our case.

Theorem 2.6 (G-Signature Theorem; [2])**.** *For g a generator of a torus in G,*

$$\begin{aligned}&\mathrm{Sign}_G\left(M^{2n}; E\right)(g)\\ &= \sum_{\nu} \left\{ 2^{t_\nu} \operatorname{ch}(E|_{M_\nu^g})(g) \hat{\mathbf{L}}(M_\nu^g) \prod_{\theta \in (0,\pi)} \prod_{j=1}^{d_\nu(\theta)} \coth \frac{x_{\nu,j}(\theta) + i\theta}{2} \right\} [M_\nu^g]\end{aligned}$$

where $t_\nu = \frac{1}{2} \dim M_\nu^g$, $d_\nu(\theta) = \dim_{\mathbb{C}} N_\nu(\theta)$, and $x_{\nu,j}(\theta)$, $1 \le j \le d_\nu(\theta)$, are the formal Chern roots of the complex bundle $N_\nu(\theta)$.

In this formula, the product over $\theta \in (0,\pi)$ is a finite product, as $d_\nu(\theta) \neq 0$ for only finitely many values of θ.

The G-index of D_E^+ evaluated at $e \in G$ is just the ordinary index of D_E^+ :

$$\operatorname{ind}\left(D_E^+\right) = \dim\ker D_E^+ - \dim\operatorname{coker} D_E^+.$$

We denote $\operatorname{ind}\left(D_E^+\right)$ by $\mathrm{Sign}(M; E)$. Index theory provides the following formula.

Theorem 2.7 ([2])**.**

$$\begin{aligned}\mathrm{Sign}_G(M^{2n}; E)(e) &= \mathrm{Sign}\left(M^{2n}; E\right)\\ &= 2^n \left(\operatorname{ch}(E)\, \hat{\mathbf{L}}(M)\right)[M].\end{aligned}$$

2.2.2. *The G-spinor-index.* For M^{2n} a spin G-manifold (i.e., a spin manifold endowed with a G-action preserving the spin structure) and E a complex G-bundle over M, [2] constructs an elliptic operator called the Dirac operator whose G-index is called the G-spinor-index and is denoted by $\mathrm{Spin}_G(M; E)$. The value of the G-spinor-index at $e \in G$ is just the ordinary index of the Dirac operator and is denoted by $\mathrm{Spin}(M; E)$.

The G-spin theorem, similar to the G-signature theorem, gives a localization formula for $\mathrm{Spin}_G(M; E)(g)$ in terms of the fixed-point data of the g-action. We now describe this result.

For a manifold X^m, the $\hat{\mathbf{A}}$-class, denoted by $\hat{\mathbf{A}}(X)$, is defined by

$$\hat{\mathbf{A}}(X) = \prod_{j=1}^{\left[\frac{m+1}{2}\right]} \frac{x_j/2}{\sinh(x_j/2)}$$

where $\left\{+x_j, -x_j | 1 \le j \le \left[\frac{m+1}{2}\right]\right\}$ are the formal Chern roots of either $TX^m \otimes \mathbb{C}$ or $(TX^m \oplus \mathbf{1}_{\mathbb{R}}) \otimes \mathbb{C}$ according to whether m is even or odd respectively. We write $\hat{A}_k(X)$ for the $4k$-dimensional component of $\hat{\mathbf{A}}(X)$.

Using the same notation and making the same assumptions as in the discussion of the G-signature theorem, we state the G-spin theorem.

Theorem 2.8 (G-Spin Theorem; [1], [2]). *For g a generator of a torus in G,*

$$\begin{aligned}&\mathrm{Spin}_G(M^{2n};E)(g)\\&=\sum_\nu\left\{\sigma_\nu\,\mathrm{ch}(E|_{M^g_\nu})(g)\hat{\mathbf{A}}\,(M^g_\nu)\prod_{\theta\in(0,\pi)}\prod_{j=1}^{d_\nu(\theta)}\frac{e^{\frac{1}{2}(i\theta-x_{\nu,j}(\theta))}}{1-e^{(i\theta-x_{\nu,j}(\theta))}}\right\}[M^g_\nu]\end{aligned}$$

where for each ν, σ_ν equals either 1 or -1 and depends on the action of g on the principal $\mathrm{Spin}\,(2n)$*-bundle associated with the spin-structure of M.*

Theorem 2.9 ([2]).

$$\begin{aligned}\mathrm{Spin}_G(M^{2n};E)(e)&=\mathrm{Spin}\left(M^{2n};E\right)\\&=2^n\left(\mathrm{ch}\,(E)\,\hat{\mathbf{A}}\,(M)\right)[M]\,.\end{aligned}$$

3. Localization theorems on certain G-manifolds

In this section, we obtain some localization results for certain "characteristic numbers" of some G-manifolds. Suppose M is an S^1-manifold. Under certain conditions, we study, in §3.1, the dependence of $\left(\hat{\mathbf{L}}(M)\cdot C\right)[M]$ on M^{S^1}. Under less restrictive conditions, we then study, in §3.2, more general numbers of the form $(P\cdot C)\,[M]$.

3.1. $\left(\hat{\mathbf{L}}(M)\cdot C\right)[M]$.

We begin with a theorem; the condition on the fixed point set appearing therein is remarked upon in §1.

Theorem 3.1. *Let G be semisimple and M^{2n} a G-manifold. Suppose there exists a circle subgroup $S\subset G$ such that every component of M^S intersects M^G. Let F_0 be the union of those components of M^S with codimensions congruent to* $0 \bmod 4$, *and let $i_0 : F_0 \hookrightarrow M$ be the inclusion map. Then $\left(\hat{\mathbf{L}}(M)\cdot C\right)[M]$ is determined by the submanifold $i_0 : F_0 \hookrightarrow M$ (and is* ***independent*** *of the S-action around F_0.) To be precise, this number can be computed in terms of $i_0^*(C)$, $\hat{\mathbf{L}}(F_0)$ and $[F_0]$ using formulae* (3.3) *and* (3.4) *and Lemma* 3.4. *In particular, if, in addition to the stated condition, $F_0=\varnothing$, then $\left(\hat{\mathbf{L}}(M)\cdot C\right)[M]=0$.*

Remark 3.2. To simplify the proof of this and some later results, we make several observations:

1. By compactness of G, a G-manifold M can always be endowed with a Riemannian structure so that the G-action on M is isometric. The tool of index theory is then readily applicable.
2. If G is a semisimple compact Lie group acting on M, then, $\widetilde{G}$, the compact universal cover of G, acts on M in a canonical way: $\widetilde{G}\to G\to \mathrm{Diff}(M)$. And there exists a circle subgroup $\widetilde{S}\subset\widetilde{G}$ with $M^S=M^{\widetilde{S}}$. Therefore, for the proof of Theorem 3.1, there is no loss of generality in assuming that G is simply connected.

Proof of Theorem 3.1. By Remark 3.2, we assume that *G is simply connected and G acts isometrically on M*. The G-action preserves the orientation of M, as G is connected. To avoid triviality, we assume that the S-action on M is nontrivial.

Let $c \in \operatorname{Im}\left(H^2(M) \to H^2(M;\mathbb{Q})\right)$. We first give a localization formula for rational numbers of the form

$$\left(\hat{\mathbf{L}}(M) \cdot c^d\right)[M]$$

in terms of the submanifold F_0.

Let $\mathcal{L}(c) \to M$ be the complex line bundle whose first Chern class is c. As G is now assumed to be simply connected, by Corollary 2.3, the G-action on M admits a (unique) lifting to a G-action on $\mathcal{L}(c)$, turning $\mathcal{L}(c)$ into a G-equivariant complex line bundle. Thus, $\operatorname{Sign}_G(M;\mathcal{L}(c))$, the $\mathcal{L}(c)$-twisted G-signature of M, is defined. Using complex coordinates z (with $|z| = 1$) on the circle subgroup S, we first develop a formula for $\operatorname{Sign}_G(M;\mathcal{L}(c))(z) \in \mathbb{C}$. As $z \in S$, $\operatorname{Sign}_G(M;\mathcal{L}(c))(z) = \operatorname{Sign}_S(M;\mathcal{L}(c))(z)$ when M is regarded as an S-manifold.

In the following, we use the notation established in §2.2.1. Let $z = e^{i\alpha}$ topologically generate S. Certainly, $M^z = M^S$. By choosing suitable orientations, the irreducible subrepresentations of the real S-module N_x $(x \in M^S)$ are of the form

$$z \mapsto \begin{pmatrix} \cos k\alpha & -\sin k\alpha \\ \sin k\alpha & \cos k\alpha \end{pmatrix}, \quad \text{or} \quad z \mapsto z^k, \text{ with } k \in \mathbb{N}.$$

Let $M^S = \bigcup_\nu M^S_\nu$, where M^S_ν are the components of M^S, and let N_ν denote the normal bundle of $M^S_\nu \hookrightarrow M$. As in §2.2.1, there is a complex bundle decomposition determined by the irreducible S-modules appearing in $(N_\nu)_x$ with $x \in M^S_\nu$:

$$N_\nu = \bigoplus_{k\in\mathbb{N}} N_\nu(k).$$

For $k \in \mathbb{N}$, let $d_\nu(k) = \dim_{\mathbb{C}} N_\nu(k)$, $t_\nu = \frac{1}{2}\dim M^S_\nu$, and $x_{\nu,j}(k)$, $j = 1, \cdots, d_\nu(k)$, be the formal Chern roots of the complex bundle $N_\nu(k)$.

According to the G-signature theorem,

$$\begin{aligned}
&\operatorname{Sign}_G(M;\mathcal{L}(c))(z) \\
&= \operatorname{Sign}_S(M;\mathcal{L}(c))(z) \\
&= \sum_\nu \left\{ 2^{t_\nu} \operatorname{ch}\left(\mathcal{L}(c)|_{M^S_\nu}\right)(z)\hat{\mathbf{L}}\left(M^S_\nu\right) \prod_{k\in\mathbb{N}} \prod_{j=1}^{d_\nu(k)} \coth\frac{x_{\nu,j}(k) + ik\alpha}{2} \right\} \left[M^S_\nu\right].
\end{aligned}$$

Letting $s_\nu(z)$ denote the νth summand, we have

$$\operatorname{Sign}_G(M;\mathcal{L}(c))(z) = \sum_\nu s_\nu(z).$$

We now seek to express $s_\nu(z)$ in terms of z instead of α.

For each component M^S_ν, there is an integer l_ν such that, for $x \in M^S_\nu$, z acts on the fiber $\mathcal{L}(c)_x$ via the complex representation $z \mapsto z^{l_\nu}$. Hence,

$$\operatorname{ch}\left(\mathcal{L}(c)|_{M^S_\nu}\right)(z) = z^{l_\nu} \cdot e^{i^*_\nu(c)}.$$

Next, observe that

$$\begin{aligned}
\coth\frac{x_{\nu,j}(k) + ik\alpha}{2} &= \frac{e^{\frac{1}{2}(x_{\nu,j}(k)+ik\alpha)} + e^{-\frac{1}{2}(x_{\nu,j}(k)+ik\alpha)}}{e^{\frac{1}{2}(x_{\nu,j}(k)+ik\alpha)} - e^{-\frac{1}{2}(x_{\nu,j}(k)+ik\alpha)}} \\
&= \frac{z^k + e^{-x_{\nu,j}(k)}}{z^k - e^{-x_{\nu,j}(k)}}.
\end{aligned}$$

Hence,

$$\begin{aligned}&\operatorname{Sign}_S\left(M;\mathcal{L}\left(c\right)\right)(z)\\&=\sum_{\nu}\left\{2^{t_\nu}z^{l_\nu}e^{i_\nu^*(c)}\hat{\mathbf{L}}\left(M_\nu^S\right)\prod_{k\in\mathbb{N}}\left(\prod_{j=1}^{d_\nu(k)}\frac{z^k+e^{-x_{\nu,j}(k)}}{z^k-e^{-x_{\nu,j}(k)}}\right)\right\}\left[M_\nu^S\right].\end{aligned}$$

So far, $s_\nu(z)$ is only defined for z a topological generator of S. If we extend $s_\nu(z)$ to a meromorphic function on $\mathbb{C}$, it then has poles at some roots of unity (certainly at 1), as can be seen by expanding $\frac{z^k+e^{-x_{\nu,j}(k)}}{z^k-e^{-x_{\nu,j}(k)}}$ into a polynomial in $x_{\nu,j}(k)$:

$$\frac{z^k+e^{-x_{\nu,j}(k)}}{z^k-e^{-x_{\nu,j}(k)}}=\frac{z^k+1}{z^k-1}+\sum_{m=1}^{t_\nu}(-1)^m\frac{2z^k}{\left(z^k-1\right)^{m+1}}\left(x_{\nu,j}(k)\right)^m.$$

However, $\operatorname{Sign}_S(M;\mathcal{L}(c))(z)$ is a well-defined complex function for all $z\in S$; indeed, since $\operatorname{Sign}_S(M;\mathcal{L}(c))\in R(S)\simeq R\left(S^1\right)$, $\operatorname{Sign}_S(M;\mathcal{L}(c))(z)$ is a Laurent polynomial in z and hence extends to an analytic function $w(z)$ on $\mathbb{C}\smallsetminus\{0\}$. In particular, $w(z)$ has no poles on the unit circle. Consequently, in $\sum_\nu s_\nu(z)$, cancellation of poles among the summands $s_\nu(z)$ must take place, implying the multitude of components in M^S if $M^S\neq\varnothing$.

The discussion of the two preceding paragraphs (in which the semisimplicity of G plays no role) applies regardless of whether $\mathcal{L}(c)$ is present or not. We summarize this discussion in a proposition about the fixed-point set of an even-dimensional S^1-manifold.

Proposition 3.3. *Let S^1 act nontrivially on an even-dimensional manifold M. Then,*

(1) *The number of components of M^{S^1} does not equal* 1.
(2) *Suppose that $M^{S^1}=\{a,b\}$, a two-point set. Then T_aM and T_bM are equivalent as S^1-representations, and* $\operatorname{Sign}(M)=0$.

Proof. If $M^{S^1}\neq\varnothing$, the poles of $s_{\nu_0}(z)$ must be cancelled by those in $\sum_{\nu\neq\nu_0}s_\nu(z)$ and hence M^{S^1} has more than one component, proving (1).

For (2), let $W_\nu=\{k\in\mathbb{N}|N_\nu(k)\neq\{0\}\}$ for $\nu=a,b$. Then the S^1-signature of M is

$$\begin{aligned}&\operatorname{Sign}_{S^1}(M)(z)\\&=\prod_{k\in W_a}\left(\frac{z^k+1}{z^k-1}\right)^{d_a(k)}[a]+\prod_{k\in W_b}\left(\frac{z^k+1}{z^k-1}\right)^{d_b(k)}[b].\end{aligned}$$

For this to be analytic on S^1, the two summands must have common poles and hence $W_a=W_b$. Let $k\in W_a=W_b$. At each kth roots of unity, the poles in the above two terms are of order $d_a(k)$ and $d_b(k)$ respectively. For the poles to cancel each other out, it must be that $d_a(k)=d_b(k)$ and one of the two points a and b receives the negative orientation. Hence, $\operatorname{Sign}(M)=\operatorname{Sign}_{S^1}(M)(z)=0$. □

Continuation of proof of Theorem 3.1. We now study the Laurent polynomial $w(z)$ $=\operatorname{Sign}_S(M;\mathcal{L}(c))(z)$ further.

We claim that $l_\nu = 0$ for all ν. For $y \in M^G$, $\mathcal{L}(c)_y$ is a 1-dimensional complex G-module. By Proposition 2.4, the semisimplicity of G implies that $\mathcal{L}(c)_y$ is a trivial G-module. Therefore, S acts trivially on $\mathcal{L}(c)_y$. By hypothesis, each component M_ν^S of M^S contains some $y \in M^G$. Hence, by discreteness of representations, S acts trivially on $\mathcal{L}(c)_x$ for every $x \in M_\nu^S$, i.e., $l_\nu = 0$.

Putting this into the formula for $s_\nu(z)$, we have

$$s_\nu(z) = \left\{ 2^{t_\nu} e^{i_\nu^*(c)} \hat{\mathbf{L}}\left(M_\nu^S\right) \prod_{k\in\mathbb{N}} \left(\prod_{j=1}^{d_\nu(k)} \frac{z^k + e^{-x_{\nu,j}(k)}}{z^k - e^{-x_{\nu,j}(k)}} \right) \right\} \left[M_\nu^S\right].$$

Regarded as a function on the extended complex plane $\bar{\mathbb{C}}$, each $s_\nu(z)$ is finite at 0 and ∞, as is clear by taking $\lim s_\nu(z)$. Then so is the Laurent polynomial $w(z) = \sum_\nu s_\nu(z)$. A Laurent polynomial which is finite at 0 and ∞ must be a constant. We now examine this constant.

$$\begin{aligned} w(0) &= \lim_{z\to 0} \sum_\nu s_\nu(z) \\ &= \sum_\nu \left(2^{t_\nu} e^{i_\nu^*(c)} \hat{\mathbf{L}}\left(M_\nu^S\right) \prod_{k\in\mathbb{N}} \prod_{j=1}^{d_\nu(k)} (-1) \right) \left[M_\nu^S\right] \\ &= \sum_\nu (-1)^{n-t_\nu}\, 2^{t_\nu} \left(e^{i_\nu^*(c)} \hat{\mathbf{L}}\left(M_\nu^S\right) \right) \left[M_\nu^S\right], \end{aligned}$$

while

$$\begin{aligned} w(\infty) &= \lim_{z\to\infty} \sum_\nu s_\nu(z) \\ &= \sum_\nu \left(2^{t_\nu} e^{i_\nu^*(c)} \hat{\mathbf{L}}\left(M_\nu^S\right) \right) \left[M_\nu^S\right]. \end{aligned}$$

Equating $w(0)$ and $w(\infty)$ yields

$$\sum_{(n-t_\nu)\equiv 1 \bmod 2} \left(2^{t_\nu} e^{i_\nu^*(c)} \hat{\mathbf{L}}\left(M_\nu^S\right) \right) \left[M_\nu^S\right] = 0.$$

Hence,

$$w(z) \equiv \sum_{\operatorname{codim} M_\nu^S \equiv 0 \bmod 4} \left(2^{t_\nu} e^{i_\nu^*(c)} \hat{\mathbf{L}}\left(M_\nu^S\right) \right) \left[M_\nu^S\right].$$

We conclude that

$$\begin{aligned} \operatorname{Sign}_G(M;\mathcal{L}(c))(e) &= \operatorname{Sign}_S(M;\mathcal{L}(c))(1) \\ &= \sum_{\operatorname{codim} M_\nu^S \equiv 0 \bmod 4} \left(2^{t_\nu} e^{i_\nu^*(c)} \hat{\mathbf{L}}\left(M_\nu^S\right) \right) \left[M_\nu^S\right]. \end{aligned} \tag{3.1}$$

Applying Theorem 2.7 to our case leads to

$$
\begin{aligned}
(3.2)\qquad & \operatorname{Sign}_G(M;\mathcal{L}(c))(e) \\
&= \operatorname{Sign}(M;\mathcal{L}(c)) \\
&= 2^n\left(\operatorname{ch}(\mathcal{L}(c))\,\hat{\mathbf{L}}(M)\right)[M] \\
&= 2^n\left(e^c\cdot\hat{\mathbf{L}}(M)\right)[M] \\
&= 2^n\left\{\left(\sum_{j=0}^{n}\frac{c^j}{j!}\right)\cdot\left(\sum_{i=0}^{[n/2]}\hat{L}_i(M)\right)\right\}[M].
\end{aligned}
$$

To obtain a formula for $\left(\hat{\mathbf{L}}(M)\cdot c^d\right)[M]$ from (3.1) and (3.2), we use a technique of [5]. First we consider the case where n is odd; let $n=2r+1$. Let λ be an integer. (3.1) and (3.2) yield

$$
\begin{aligned}
&\left(\sum_{j=0}^{r}\frac{1}{(2j+1)!}(\lambda c)^{2j+1}\hat{L}_{r-j}(M)\right)\left[M^{4r+2}\right] \\
&= 2^{-n}\operatorname{Sign}\left(M^{2n};\mathcal{L}(\lambda c)\right) \\
&= 2^{-n}\sum_{\operatorname{codim}M_\nu^S\equiv 0 \bmod 4}\left(2^{t_\nu}e^{i_\nu^*(\lambda c)}\hat{\mathbf{L}}\left(M_\nu^S\right)\right)\left[M_\nu^S\right].
\end{aligned}
$$

By letting $\lambda=1,2,\cdots,r+1$, the above is a system of $(r+1)$ linear equations in the $(r+1)$ variables $\left(\frac{1}{(2j+1)!}c^{2j+1}\hat{L}_{r-j}(M)\right)[M]$, $j=0,1,\cdots,r$. The coefficient matrix is

$$
A=(a_{ij})_{i,j=1}^{r+1}=\left(i^{2j-1}\right)_{i,j=1}^{r+1},
$$

and

$$
\begin{aligned}
\det A &= \det\left(\left(i^{2j-1}\right)_{i,j=1}^{r+1}\right) \\
&= \det\left(i\cdot(i^2)^{j-1}\right) \\
&= [(r+1)!]\det\left(\left(i^2\right)^{j-1}\right) \\
&= [(r+1)!]\prod_{1\le\alpha<\beta\le r+1}\left(\beta^2-\alpha^2\right) \\
&\neq 0.
\end{aligned}
$$

Let $A^{-1}=(b_{ij})_{i,j=1}^{r+1}$. Solving the linear system gives

$$
\begin{aligned}
(3.3)\qquad & \left(c^{2j+1}\hat{L}_{r-j}(M)\right)\left[M^{4r+2}\right] \\
&= \frac{(2j+1)!}{2^{4r+2}}\sum_{\lambda=1}^{r+1}\left\{b_{(j+1),\lambda}\cdot\sum_{\operatorname{codim}M_\nu^S\equiv 0 \bmod 4}\left(2^{t_\nu}e^{i_\nu^*(\lambda c)}\hat{\mathbf{L}}\left(M_\nu^S\right)\right)\left[M_\nu^S\right]\right\}.
\end{aligned}
$$

The case where n is even is similarly handled. Let $n = 2r$. Let λ be an integer. (3.1) and (3.2) yield

$$\begin{aligned}&\left(\sum_{j=0}^{r} \frac{1}{(2j)!} (\lambda c)^{2j} \hat{L}_{r-j}(M)\right)\left[M^{4r}\right]\\&= 2^{-n}\operatorname{Sign}\left(M^{2n}; \mathcal{L}(\lambda c)\right)\\&= 2^{-n} \sum_{\operatorname{codim} M_\nu^S \equiv 0 \bmod 4} \left(2^{t_\nu} e^{i_\nu^*(\lambda c)} \hat{\mathbf{L}}\left(M_\nu^S\right)\right)\left[M_\nu^S\right].\end{aligned}$$

By letting $\lambda = 1, 2, \cdots, r+1$, the above is a system of $(r+1)$ linear equations in the $(r+1)$ variables $\left(\frac{1}{(2j)!} c^{2j} \hat{L}_{r-j}(M)\right)[M]$, $j = 0, 1, \cdots, r$. The coefficient matrix is

$$A = (a_{ij})_{i,j=1}^{r+1} = \left(i^{2j-2}\right)_{i,j=1}^{r+1},$$

and

$$\begin{aligned}\det A &= \det\left(\left(i^{2j-2}\right)_{i,j=1}^{r+1}\right)\\&= \det\left((i^2)^{j-1}\right)\\&= \prod_{1 \le \alpha < \beta \le r+1} \left(\beta^2 - \alpha^2\right)\\&\ne 0.\end{aligned}$$

Let $A^{-1} = (b_{ij})_{i,j=1}^{r+1}$. Solving the linear system gives

$$\begin{aligned}&\left(c^{2j} \hat{L}_{r-j}(M)\right)\left[M^{4r}\right]\\&= \frac{(2j)!}{2^{4r}} \sum_{\lambda=1}^{r+1} \left\{ b_{(j+1),\lambda} \cdot \sum_{\operatorname{codim} M_\nu^S \equiv 0 \bmod 4} \left(2^{t_\nu} e^{i_\nu^*(\lambda c)} \hat{\mathbf{L}}\left(M_\nu^S\right)\right)\left[M_\nu^S\right] \right\}.\end{aligned} \tag{3.4}$$

For $c \in \operatorname{Im}\left(H^2(M) \to H^2(M; \mathbb{Q})\right)$, (3.3) and (3.4) give localization formulae for the rational number

$$\left(\hat{\mathbf{L}}(M) \cdot c^d\right)[M]$$

in terms of the submanifold $F_0 \subset M$ consisting of those components of M^S with codimensions congruent to 0 mod 4.

To finally conclude the proof, we need to allow product of various $c_j \in H^2(M; \mathbb{Q})$ to occur in the formula. Consider

$$\left(\hat{\mathbf{L}}(M) \cdot \prod\nolimits_j c_j^{d_j}\right)[M].$$

Let $\sum_j d_j = d$. By Lemma 3.4 (stated below and to be established in §3.3), $\prod_j c_j^{d_j} = \sum_i r_i x_i^d$ for some $x_i \in H^2(M; \mathbb{Q})$ and $r_i \in \mathbb{Q}$. Hence

$$\begin{aligned}&\left(\hat{\mathbf{L}}(M) \cdot \prod\nolimits_j c_j^{d_j}\right)[M]\\&= \sum_i r_i \left(\hat{\mathbf{L}}(M) \cdot x_i^d\right)[M].\end{aligned}$$

Since each $\left(\hat{\mathbf{L}}(M) \cdot x_i^d\right)[M]$ is computable, using equations (3.3) and (3.4), in terms of the submanifold $F_0 \subset M$ consisting of those components of M^S with codimensions congruent to $0 \bmod 4$, so is $\left(\hat{\mathbf{L}}(M) \cdot \prod_j c_j^{d_j}\right)[M]$.

This concludes the proof of Theorem 3.1. □

We now state the lemma we needed at the end of the proof of Theorem 3.1.

Lemma 3.4. *Suppose $\mathcal{A} = \bigoplus_{i=0}^{\infty} A_i$ is a graded $\mathbb{k}$-algebra, $\mathbb{k}$ being a field of characteristic 0. Let l be a given positive integer, let $\mathcal{B} = \bigoplus_{i=0}^{\infty} B_i \subset \bigoplus_{i=0}^{\infty} A_{il}$ be the subalgebra of $\mathcal{A}$ generated by A_l, and let $V_d = \operatorname{Span}\{a^d | a \in A_l\}$, $d \in \mathbb{N}$. Then $B_d \subseteq V_d$.*

We relegate the proof to §3.3.

Theorem 3.1 may be restated as follows:

Corollary 3.5. *Let M^{2n} be an S^1-manifold. Suppose, for some semisimple Lie group G, the S^1-action extends to a G-action such that every component of M^{S^1} intersects M^G. Let F_0 be the union of those components of M^{S^1} with codimensions congruent to $0 \bmod 4$, and let $i_0 : F_0 \hookrightarrow M$ be the inclusion map. Then $\left(\hat{\mathbf{L}}(M) \cdot C\right)[M]$ is determined by the submanifold $i_0 : F_0 \hookrightarrow M$ (and is* ***independent*** *of the S^1-action around F_0.) In particular, if, in addition to the stated condition, $F_0 = \varnothing$, then $\left(\hat{\mathbf{L}}(M) \cdot C\right)[M] = 0$.*

3.2. $(P \cdot C)[M]$**.** We now consider rational numbers of the form $(P \cdot C)[M^{2n}]$. The main results of this subsection are Theorems 3.6 and 3.11.

Theorem 3.6. *Let G be semisimple and let M^{2n} be a G-manifold. Suppose there exists a circle subgroup $S \subset G$ such that every component of M^S intersects M^G. Then $(P \cdot C)[M]$ can be determined in terms of the submanifold $i : M^S \hookrightarrow M$ and the S-action on the normal bundle N of $M^S \hookrightarrow M$. To be precise, this number can be computed in terms of $i^*(C)$, $p\left(M^S\right)$ (the total Pontrjagin class of M^S), $\left[M^S\right]$, and, the action of S on N.*

The corresponding vanishing theorem is much simpler to state:

Corollary 3.7. *Let G be semisimple and let M^{2n} be a G-manifold. If, for some $g \in G$, $M^g = \varnothing$, then $(P \cdot C)[M] = 0$.*

Proof. If $M^g = \varnothing$, then $M^T = \varnothing$, where T is some maximal torus containing g. As a standard fact about torus-actions, there always exists a circle subgroup $S \subset T$ such that $M^S = M^T$. Using such an S and applying Theorem 3.6 yields the desired result. □

In Corollary 4.9, this result will be considerably strengthened.

We now prepare for the proof of Theorem 3.6.

As before, endow M^{2n} with a Riemannian structure so that G acts isometrically. Let $h : M^{2n} \to BSO(2n)$ be the classifying map for the tangent bundle TM. Define

$$\mathfrak{b} : RSO\,(2n) \longrightarrow K\,(BSO\,(2n))$$

by

$$\mathfrak{b} : V \mapsto ESO\,(2n) \times_{SO(2n)} V.$$

$\mathfrak{b}$ "bundlizes" a complex $SO(2n)$-module. On elements of $K(M)$ associated with the Riemannian structure, i.e., on elements in $(h^* \circ \mathfrak{b})(RSO(2n))$, the Chern character admits a universal interpretation as a ring homomorphism $\mathfrak{Ch}$ on the complex $SO(2n)$-representations to the cohomology of $BSO(2n)$:

$$\mathfrak{Ch} : RSO(2n) \xrightarrow{\mathfrak{b}} K(BSO(2n)) \xrightarrow{\mathrm{ch}} H^*(BSO(2n);\mathbb{Q}).$$

Let σ_d be the elementary symmetric polynomial of degree d in n variables. Let $x_1, \cdots, x_n$ be the basic characters of the maximal torus T of $SO(2n)$, regarded as cohomology classes in $H^2(BT;\mathbb{Q})$. As usual, we identify $H^*(BSO(2n);\mathbb{Q})$ with $H^*(BT;\mathbb{Q})^W$, the subring of $H^*(BT;\mathbb{Q})$ invariant under the Weyl group W.

Lemma 3.8 ([2])**.** *There exists $U_{\sigma_d} \in RSO(2n)$ such that*

$$\mathfrak{Ch}(U_{\sigma_d}) = \sigma_d\left(x_1^2, \cdots, x_n^2\right) + \textit{higher dimensional terms.}$$

Those higher dimensional terms do not involve the Euler class $\prod_{i=1}^n x_i$.

We now use this result to immediately deduce:

Lemma 3.9. *Let $P_d(p_1, \cdots, p_d) \in H^{4d}\left(M^{2n}\right)$ be a homogeneous $4d$-dimensional integral polynomial in the first d Pontrjagin classes $p_1, \cdots, p_d$ of a G-manifold M^{2n}. Then there exists $E(P_d) \in K_G(M)$ with $\mathrm{ch}\, E(P_d) = P_d(p_1, \cdots, p_d)$.*

Proof. Let $r = \left[\frac{n}{2}\right]$. We apply a finite induction on d.

First we consider the case where $d = r$. Let $\tilde{U}_{\sigma_i} = \mathfrak{b}(U_{\sigma_i}) \in K(BSO(2n))$, and let

$$E(P_r) = h^*\left(P_r\left(\tilde{U}_{\sigma_1}, \cdots, \tilde{U}_{\sigma_r}\right)\right) \in K(M).$$

Note that, because h classifies the principal $SO(2n)$-bundle, τ, associated with the Riemannian structure of M, $E(P_r) = \tau \times_{SO(2n)} P_r(U_{\sigma_1}, \cdots, U_{\sigma_r})$. Hence, $E(P_r)$ admits a G-action induced by the G-action on τ, i.e., $E(P_r) \in K_G(M)$. Then

$$\begin{aligned}
\mathrm{ch}\, E(P_r) &= \mathrm{ch}\, h^*\left(P_r\left(\tilde{U}_{\sigma_1}, \cdots, \tilde{U}_{\sigma_r}\right)\right) \\
&= h^* \,\mathrm{ch}\, \mathfrak{b}\left(P_r(U_{\sigma_1}, \cdots, U_{\sigma_r})\right) \\
&= h^* \mathfrak{Ch}\left(P_r(U_{\sigma_1}, \cdots, U_{\sigma_r})\right) \\
&= h^* P_r\left(\mathfrak{Ch}U_{\sigma_1}, \cdots, \mathfrak{Ch}U_{\sigma_r}\right) \\
&= h^*\left(P_r(\sigma_1, \cdots, \sigma_r) + \text{higher dimensional terms}\right) \\
&= P_r(h^*\sigma_1, \cdots, h^*\sigma_r) + \text{higher dimensional terms} \\
&= P_r(p_1, \cdots, p_r) + \text{higher dimensional terms} \\
&= P_r(p_1, \cdots, p_r)
\end{aligned}$$

where the "higher dimensional terms" are of dimension at least $4(r+1) > 2n$ and hence are 0.

Next we examine the case where $d = r - 1$. Let

$$E'(P_{r-1}) = h^*\left(P_{r-1}\left(\tilde{U}_{\sigma_1}, \cdots, \tilde{U}_{\sigma_{r-1}}\right)\right) \in K(M).$$

Then

$$\begin{aligned}
\mathrm{ch}\, E'(P_{r-1}) &= P_{r-1}(p_1, \cdots, p_{r-1}) + \text{higher dimensional terms} \\
&= P_{r-1}(p_1, \cdots, p_{r-1}) + Q_r(p_1, \cdots, p_r)
\end{aligned}$$

where $Q_r(p_1, \cdots, p_r)$ is a homogeneous $4r$-dimensional integral polynomial. Simply let $E(P_{r-1}) = E'(P_{r-1}) - E(Q_r)$; then

$$\begin{aligned}\operatorname{ch} E(P_{r-1}) &= \operatorname{ch}\left(E'(P_{r-1}) - E(Q_r)\right)\\ &= \operatorname{ch} E'(P_{r-1}) - \operatorname{ch} E(Q_r)\\ &= P_{r-1}(p_1, \cdots, p_{r-1}).\end{aligned}$$

Continuing in this fashion until reaching the case where $d = 0$ proves the result. □

Since $\operatorname{Sign}_G(M; E)$ is additive in E, E can be allowed to be an element of $K_G(M)$.

Proof of Theorem 3.6. By Remark 3.2, we assume that G *is simply connected and* G *acts isometrically on* M. The G-action preserves the orientation of M, as G is connected.

By Lemma 3.4, it suffices to prove the result for $\left\{c^{n-2d} \cdot P_d\right\}[M^{2n}]$ where $c \in \operatorname{Im}\left(H^2(M) \to H^2(M; \mathbb{Q})\right)$ and $P_d \in H^{4d}\left(M^{2n}; \mathbb{Q}\right)$ is a homogeneous $4d$-dimensional polynomial in the Pontrjagin classes of M^{2n}.

Let $\mathcal{L}(c) \to M$ be the complex line bundle whose first Chern class is c. Following the same reasoning as in the proof of Theorem 3.1, the G-action on M admits a (unique) lifting to a G-action on $\mathcal{L}(c)$ and S acts trivially on $\mathcal{L}(c)_x$ for every $x \in M_\nu^S$.

Construct $E(P_d)$ as in Lemma 3.9. We consider $\operatorname{Sign}_G(M; \mathcal{L}(c) \otimes E(P_d))(z)$ for $z \in S$. We use the same notation as in the proof of Theorem 3.1. Let z be (the complex S^1-coordinate of) any topological generator of S. Then $M^z = M^S$. Calculating just as in the proof of Theorem 3.1, we have

$$\begin{aligned}&\operatorname{Sign}_G(M; \mathcal{L}(c) \otimes E(P_d))(z)\\ &= \operatorname{Sign}_S(M; \mathcal{L}(c) \otimes E(P_d))(z)\\ &= \sum_\nu 2^{t_\nu} \operatorname{ch}\left(\mathcal{L}(c) \otimes E(P_d)|_{M_\nu^S}\right)(z)\hat{\mathbf{L}}\left(M_\nu^S\right) \prod_{k\in\mathbb{N}} \prod_{j=1}^{d_\nu(k)} \frac{z^k + e^{-x_{\nu,j}(k)}}{z^k - e^{-x_{\nu,j}(k)}}\left[M_\nu^S\right]\\ &= \sum_\nu 2^{t_\nu} e^{i_\nu^*(c)} z^{l_\nu} \operatorname{ch}\left(E(P_k)|_{M_\nu^S}\right)(z)\hat{\mathbf{L}}\left(M_\nu^S\right) \prod_{k\in\mathbb{N}} \prod_{j=1}^{d_\nu(k)} \frac{z^k + e^{-x_{\nu,j}(k)}}{z^k - e^{-x_{\nu,j}(k)}}\left[M_\nu^S\right]\\ &= \sum_\nu 2^{t_\nu} e^{i_\nu^*(c)} \operatorname{ch}\left(E(P_k)|_{M_\nu^S}\right)(z)\hat{\mathbf{L}}\left(M_\nu^S\right) \prod_{k\in\mathbb{N}} \prod_{j=1}^{d_\nu(k)} \frac{z^k + e^{-x_{\nu,j}(k)}}{z^k - e^{-x_{\nu,j}(k)}}\left[M_\nu^S\right].\end{aligned}$$

The complex cohomology class

$$\operatorname{ch}\left(E(P_d)|_{M_\nu^S}\right)(z)$$

by definition is determined by the submanifold $M_\nu^S \overset{i_\nu}{\hookrightarrow} M$ and the S-action around it. Hence the above formula shows that the complex number

$$\operatorname{Sign}_G(M; \mathcal{L}(c) \otimes E(P_d))(z)$$

is determined by the data of the submanifold $i : M^S \hookrightarrow M$ and the S-action on N. By continuity of index as a character function on G, the integer

$$\begin{aligned}\operatorname{Sign}(M;\mathcal{L}(c)\otimes E(P_d)) &= \operatorname{Sign}_G(M;\mathcal{L}(c)\otimes E(P_d))(e)\\ &= \lim_{z\to 1}\operatorname{Sign}_S(M;\mathcal{L}(c)\otimes E(P_d))(z)\end{aligned}$$

is also determined by the data of the submanifold $i : M^S \hookrightarrow M$ and the S-action on N. Let $2n = 4r$ or $4r+2$. An explicit formula for this limit when $d = r$ is to be given in Remark 3.10; when $d < r$, the limit can be calculated using an induction shown below.

Using Theorem 2.7, we have

$$\begin{aligned}&\operatorname{Sign}(M;\mathcal{L}(c)\otimes E(P_d))\\ &= 2^n\left\{\operatorname{ch}(\mathcal{L}(c)\otimes E(P_d))\cdot \hat{\mathbf{L}}(M)\right\}[M]\\ &= 2^n\left\{e^c\cdot P_d\cdot \hat{\mathbf{L}}(M)\right\}[M].\end{aligned}$$

We apply a finite induction on d. First we examine the case where $d = r$. Then,

$$\begin{aligned}&\operatorname{Sign}(M;\mathcal{L}(c)\otimes E(P_r))\\ &= 2^n\left\{e^c\cdot P_r\cdot \hat{\mathbf{L}}(M)\right\}\left[M^{2n}\right]\\ &= \begin{cases} 2^{2r}P_r\left[M^{4r}\right] & \text{when } 2n = 4r,\\ 2^{2r+1}\left\{c\cdot P_r\right\}\left[M^{4r+2}\right] & \text{when } 2n = 4r+2,\end{cases}\end{aligned}$$

proving the result for $d = r$.

Next we examine the case where $d = r-1$. Then

$$\begin{aligned}&\operatorname{Sign}(M;\mathcal{L}(c)\otimes E(P_{r-1}))\\ &= 2^n\left\{\left(\sum_{j=0}^{n}\frac{c^j}{j!}\right)\cdot P_{r-1}\cdot\left(\sum_{i=0}^{r}\hat{L}_i(M)\right)\right\}\left[M^{2n}\right]\\ &= \begin{cases} 2^{2r}\left\{\frac{c^2}{2!}\cdot P_{r-1} + P_{r-1}\cdot \hat{L}_1\right\}\left[M^{4r}\right] & \text{when } 2n = 4r,\\ 2^{2r+1}\left\{\frac{c^3}{3!}\cdot P_{r-1} + c\cdot P_{r-1}\cdot \hat{L}_1\right\}\left[M^{4r+2}\right] & \text{when } 2n = 4r+2.\end{cases}\end{aligned}$$

Note that $P_{r-1}\cdot \hat{L}_1(M)$ is a homogeneous $4r$-dimensional polynomial in the Pontrjagin classes, and thus, by the proven case (in which $d = r$),

$$\left\{P_{r-1}\cdot \hat{L}_1(M^{4r})\right\}\left[M^{4r}\right]$$

and

$$\left\{c\cdot P_{r-1}\cdot \hat{L}_1(M^{4r+2})\right\}\left[M^{4r+2}\right]$$

can be calculated by localization. Hence, so can

$$\left\{c^2\cdot P_{r-1}\right\}\left[M^{4r}\right]$$

and

$$\left\{c^3\cdot P_{r-1}\right\}\left[M^{4r+2}\right]$$

be calculated by localization.

Continuing in this fashion until reaching the case where $d = 0$ completes the induction. □

Remark 3.10. We now indicate how to calculate $P_r\left[M^{4r}\right]$ or $(c \cdot P_r)\left[M^{4r+2}\right]$ using fixed-point data, which is our first inductive step in the proof of Theorem 3.6.

Observe that when $\dim M = 4r$, the localization result for $P_r[M]$ requires neither the semisimplicity of G nor the relation between M^{S^1} and M^G because $\mathcal{L}(c)$ does not figure in the calculation. For this case, an explicit localization formula is given in [2]. We first briefly describe this formula; we then indicate one for $(c \cdot P_r)[M]$ when $\dim M = 4r + 2$.

In light of the splitting principle, $P_r(p_1, \cdots, p_r)$ may be formally regarded as a polynomial $P'_r\left(x_1^2, \cdots, x_n^2\right)$ symmetric in $x_1^2, \cdots, x_n^2$ (where $x_1, \cdots, x_n$ can be regarded as the basic characters of the maximal torus T of $SO(2n)$). We will write the formulae in terms of P'_r.

First let $\dim M = 2n = 4r$. S^1 acts on the normal bundle N_ν with weights (counting multiplicity) $\omega_{\nu,j} \in \mathbb{Z}$, $j = 1, \cdots, n - t_\nu$; let $y_{\nu,j}$ be the corresponding Chern roots. (In detail, one constructs a "flag bundle" $P_\nu \xrightarrow{\pi} M_\nu^{S^1}$ such that $\pi^* N_\nu = \bigoplus_j \eta_{\nu,j}$, each $\eta_{\nu,j}$ being an S^1-invariant complex line bundle on which $z \in S^1$ acts via multiplication by $z^{\omega_{\nu,j}}$; and $y_{\nu,j} = c_1(\eta_{\nu,j})$.)

Using index theory, [2] shows that, formally,

$$
\begin{aligned}
&P_r(p_1, \cdots, p_r)\left[M^{4r}\right] \\
&= \frac{1}{2^n} \lim_{z \to 1} \mathrm{Sign}_{S^1}(M; E(P_r))(z) \\
&= \sum_\nu \frac{P'_r\left(x_1'^2, \cdots, x_{t_\nu}'^2; (y_{\nu,1} + i\omega_{\nu,1})^2, \cdots, (y_{\nu,n-t_v} + i\omega_{\nu,n-t_v})^2\right)}{\prod_{j=1}^{n-t_v}(y_{\nu,j} + i\omega_{\nu,j})}\left[M_\nu^{S^1}\right]
\end{aligned}
$$

where $x_1'^2, \cdots, x_{t_\nu}'^2$ are the formal "Pontrjagin roots" of $M_\nu^{S^1}$.

Using a calculation identical to the one which leads [2] to the above formula, one easily obtains, when $\dim M = 2n = 4r + 2$, that

$$
\begin{aligned}
&\{c \cdot P_r(p_1, \cdots, p_r)\}\left[M^{4r+2}\right] \\
&= \frac{1}{2^n} \lim_{z \to 1} \mathrm{Sign}_{S^1}(M; \mathcal{L}(c) \otimes E(P_r))(z) \\
&= \sum_\nu \frac{c \cdot P'_r\left(x_1'^2, \cdots, x_{t_\nu}'^2; (y_{\nu,1} + i\omega_{\nu,1})^2, \cdots, (y_{\nu,n-t_v} + i\omega_{\nu,n-t_v})^2\right)}{\prod_{j=1}^{n-t_v}(y_{\nu,j} + i\omega_{\nu,j})}\left[M_\nu^{S^1}\right].
\end{aligned}
$$

Using these formulae and the induction steps shown in the proof of Theorem 3.6, one can then calculate $\left\{c^{n-2d} \cdot P_d(p_1, \cdots, p_d)\right\}[M^{2n}]$ by localization.

In (the proofs of) Theorem 3.1 and 3.6, the determination of certain "characteristic numbers" involving c is seen to require no knowledge of the lifted G-action on $\mathcal{L}(c)$; this relies on the fact that the G-action on $\mathcal{L}(c)|_{M^S}$ is trivial, a consequence of the semisimplicity of G and the condition on the relation between M^S and M^G.

In general, when $M^S = \varnothing$, to deduce the vanishing of the numbers in question, we *only* need to know the existence of a lifting of the G-action to an action on $\mathcal{L}(c)$ *without* having to know how to produce one; Corollaries 2.2 and 2.3 give some sufficient conditions for the existence of a lifting. When $M^S \neq \varnothing$, a "localization" result on the numbers in question will require the knowledge of a lifting. We summarize this discussion in the following theorem.

Theorem 3.11. *Given a G-manifold M, let $c_j \in \operatorname{Im}\left(H^2(M) \to H^2(M;\mathbb{Q})\right)$ and $R = \left(P \cdot \prod_j c_j\right)[M]$.*

*If **either** $\pi_1(G) = \{0\}$ **or** $H^1(M) = \{0\}$, then, given any circle subgroup $S \subset G$, R can be determined in terms of: the submanifold $M^S \hookrightarrow M$, the S-action around M^S, and the restriction to $\mathcal{L}(c_j)|_{M^S}$ of **any** lifted S-action on each $\mathcal{L}(c_j)$.*

If G is semisimple, then by regarding M as a $\widetilde{G}$-manifold where $\widetilde{G}$ is the universal cover of G, R can be similarly localized as in the previous case.

In either case, if $M^g = \varnothing$ for some $g \in G$, then $R = 0$.

This is clear from the calculations we have done and Corollary 2.2.

We now use the last result to make an observation. [7] shows that if X is a T-space (T being a torus) with vanishing even-dimensional rational homotopy groups, then X^T is either empty or connected. Proposition 3.3 shows that if M is an even-dimensional manifold admitting a nontrivial T-action, then M^T is either empty or disconnected. Hence,

Corollary 3.12. *Let M be an even-dimensional manifold satisfying $H^1(M) = \{0\}$ and $\pi_{2k}(M) \otimes \mathbb{Q} = \{0\}$ for all $k \in \mathbb{N}$. If, for some classes $c_j \in H^2(M;\mathbb{Q})$ and some rational polynomial P in the Pontrjagin classes of M, $\left(P \cdot \prod_j c_j\right)[M] \neq 0$, then M admits no nontrivial torus-action.*

Proof. Suppose M admits a T-action, T being some torus. As $H^1(M) = \{0\}$ and $\left(P \cdot \prod_j c_j\right)[M] \neq 0$, $M^T \neq \varnothing$ by Theorem 3.11. As $\pi_{2k}(M) \otimes \mathbb{Q} = \{0\}$, M^T must be connected by the result of [7] quoted above; as $\dim M$ is even, M^T must be disconnected by Proposition 3.3. This contradiction proves the result. □

This implies a result of [12], which shows that, when M is as above, the nonvanishing of $\left(\prod_j c_j\right)[M]$ is an obstruction to S^1-actions on M. An example is yet to be seen in which Corollary 3.12 is genuinely stronger.

3.3. **Proof of Lemma 3.4.** We now prove Lemma 3.4.

Lemma 3.13. *Suppose $\mathcal{A} = \bigoplus_{i=0}^{\infty} A_i$ is a graded $\Bbbk$-algebra, $\Bbbk$ being a field of characteristic 0. Let l be a given positive integer, let $\mathcal{B} = \bigoplus_{i=0}^{\infty} B_i \subset \bigoplus_{i=0}^{\infty} A_{il}$ be the subalgebra of $\mathcal{A}$ generated by A_l, and let $V_d = \operatorname{Span}\{a^d | a \in A_l\}$, $d \in \mathbb{N}$. Then $B_d \subseteq V_d$.*

Proof. What needs to be shown is the fact that an element in B_d of the form

$$\prod_{j=1}^{m} c_j^{d_j}, \text{ with } c_j \in A_l,\ d_j > 0, \text{ and } \sum_{j=1}^{m} d_j = d \tag{3.5}$$

can be written as a linear combination of elements of the form a^d with $a \in A_l$.

We apply induction on the number m of distinct A_l-factors in (3.5). The case of $m = 1$ is automatic. Assuming the result for all $m < N$, it is then to be shown that it holds for $m = N$.

In what follows, we adopt the multi-index notation: $\xi = (\xi_1, \cdots, \xi_N)$ where ξ_j are nonnegative integers, $|\xi| = \sum_{j=1}^{N} \xi_j$, $\mathbf{c} = (c_1, \cdots, c_N)$ where $c_j \in A_l$, $\mathbf{c}^\xi = \prod_{j=1}^{N} c_j^{\xi_j}$, M_ξ are the multinomial coefficients.

Let $S(\mathbf{c}) = (c_1 + \cdots + c_N)^d$. Consider the multinomial expansion of $S(\mathbf{c})$:

$$S(\mathbf{c}) = S_0(\mathbf{c}) + T(\mathbf{c}),$$

where

$$S_0(\mathbf{c}) = \sum_{|\xi|=d, \text{ and } \xi_j=0 \text{ for some } j} M_\xi \cdot \mathbf{c}^\xi$$

and

$$T(\mathbf{c}) = \sum_{|\xi|=d, \text{ and } \xi_j>0 \text{ for all } j} M_\xi \cdot \mathbf{c}^\xi.$$

Our task is to show that *each* summand in $T(\mathbf{c})$ can be written as a linear combination of elements of the form a^d with $a \in A_l$.

In $S_0(\mathbf{c})$, each summand contains fewer than N distinct A_l-factors; thus, by the induction hypothesis, $S_0(\mathbf{c}) \in V_d$. Since $T(\mathbf{c}) = (\sum c_j)^d - S_0(\mathbf{c})$, $T(\mathbf{c}) \in V_d$. We now examine $T(\mathbf{c})$ more closely:

$$\begin{aligned} T(\mathbf{c}) &= \sum_{\xi_1=1}^{d-(N-1)} \left(c_1^{\xi_1} \cdot \sum_{\substack{\xi_2+\cdots+\xi_N=d-\xi_1 \\ \xi_j>0}} \left(M_\xi \prod_{j=2}^{N} c_j^{\xi_j} \right) \right) \\ &= \sum_{\xi_1=1}^{d-(N-1)} \left\{ c_1^{\xi_1} \cdot \sum_{\xi_2=1}^{d-(N-2)-\xi_1} \left(c_2^{\xi_2} \cdot \sum_{\substack{\xi_3+\cdots+\xi_N=d-(\xi_1+\xi_2) \\ \xi_j>0}} \left(M_\xi \prod_{j=3}^{N} c_j^{\xi_j} \right) \right) \right\} \\ &= \cdots . \end{aligned}$$

For each $i \in \{1, \cdots, (N-1)\}$, let

$$T(\mathbf{c}; \xi_1, \cdots, \xi_i) = \sum_{\substack{\xi_{i+1}+\cdots+\xi_N=d-(\xi_1+\cdots+\xi_i) \\ \xi_j>0}} \left(M_\xi \prod_{j=i+1}^{N} c_j^{\xi_j} \right).$$

We then can rewrite the above as:

$$\begin{aligned} T(\mathbf{c}) &= \sum_{\xi_1=1}^{d-(N-1)} \left(c_1^{\xi_1} T(\mathbf{c}; \xi_1) \right) \\ &= \sum_{\xi_1=1}^{d-(N-1)} \left\{ c_1^{\xi_1} \cdot \sum_{\xi_2=1}^{d-(N-2)-\xi_1} \left(c_2^{\xi_2} T(\mathbf{c}; \xi_1, \xi_2) \right) \right\} \\ &= \cdots . \end{aligned} \tag{3.6}$$

Let $\mathbf{c}_{(i,t)} = (c_1, \cdots, tc_i, \cdots, c_N)$ be the N-tuple obtained from $\mathbf{c}$ by replacing the ith entry c_i with tc_i. Consider the system of linear equations in $\left(c_1^{\xi_1} T(\mathbf{c}; \xi_1) \right)$, $1 \le \xi_1 \le d - (N-1)$:

$$\sum_{\xi_1=1}^{d-(N-1)} t^{\xi_1} \left(c_1^{\xi_1} T(\mathbf{c}; \xi_1) \right) = T\left(\mathbf{c}_{(1,t)}\right), \quad t = 1, \cdots, d-(N-1).$$

The coefficients of this linear system form a Vandermonde matrix, hence this system is invertible. As $T\left(\mathbf{c}_{(1,t)}\right) \in V_d$ for each t, we deduce that

$$\left(c_1^{\xi_1} T(\mathbf{c};\xi_1)\right) \in V_d, \quad \text{for } 1 \leq \xi_1 \leq d-(N-1). \tag{3.7}$$

By (3.6), for each ξ_1,

$$\begin{aligned} c_1^{\xi_1} T(\mathbf{c};\xi_1) &= c_1^{\xi_1} \cdot \sum_{\xi_2=1}^{d-(N-2)-\xi_1} \left(c_2^{\xi_2} \cdot T\left(\mathbf{c};\xi_1,\xi_2\right)\right) \\ &= \sum_{\xi_2=1}^{d-(N-2)-\xi_1} \left(c_1^{\xi_1} c_2^{\xi_2} T\left(\mathbf{c};\xi_1,\xi_2\right)\right). \end{aligned}$$

Given ξ_1, consider the following linear system in $c_1^{\xi_1} c_2^{\xi_2} T\left(\mathbf{c};\xi_1,\xi_2\right)$:

$$\sum_{\xi_2=1}^{d-(N-2)-\xi_1} t^{\xi_2} c_1^{\xi_1} c_2^{\xi_2} T\left(\mathbf{c};\xi_1,\xi_2\right) = c_1^{\xi_1} T(\mathbf{c}_{(2,t)};\xi_1),\ t=1,\cdots,d-(N-2)-\xi_1.$$

The coefficients of this linear system form a Vandermonde matrix, hence this system is invertible. By (3.7), $c_1^{\xi_1} T(\mathbf{c}_{(2,t)};\xi_1) \in V_d$ for each t, from which we deduce that

$$c_1^{\xi_1} c_2^{\xi_2} T\left(\mathbf{c};\xi_1,\xi_2\right) \in V_d, \text{ for } 1 \leq \xi_1 \leq d-(N-1) \text{ and } 1 \leq \xi_2 \leq d-(N-2)-\xi_1.$$

Continuing this argument, we would arrive finally at

$$\left(\prod_{j=1}^{N-1} c_j^{\xi_j}\right) \cdot T\left(\mathbf{c};\xi_1,\cdots,\xi_{N-1}\right) \in V_d.$$

By definition,

$$\begin{aligned} T\left(\mathbf{c};\xi_1,\cdots,\xi_{N-1}\right) &= \sum_{\substack{\xi_N=d-(\xi_1+\cdots+\xi_{N-1}) \\ \xi_j>0}} \left(M_\xi \prod_{j=N}^{N} c_j^{\xi_j}\right) \\ &= M_\xi c_N^{d-(\xi_1+\cdots+\xi_{N-1})}. \end{aligned}$$

Hence, for $|\xi| = d$,

$$\left(\prod_{j=1}^{N} c_j^{\xi_j}\right) \in V_d$$

as desired. □

4. Obstructions from the fundamental group

In this section, we study obstructions, stemming from $\pi_1(M)$, to the existence of certain group actions on M. The results produced here are "higher" versions of those in §3 in the following sense.

Given a manifold M, an element $x \in H^*(\pi;\mathbb{Q})$ determines a rational characteristic number $\left(f^*(x) \cdot \hat{\mathbf{L}}(M)\right)[M]$, which S. P. Novikov calls "a higher signature of M". [3] calls the number $\left(f^*(x) \cdot \hat{\mathbf{A}}(M)\right)[M]$ "a higher $\hat{\mathbf{A}}$-genus of M".

In this section, we consider, for certain $x \in H^*(\pi;\mathbb{Q})$, numbers of the form $(f^*(x) \cdot P \cdot C)[M]$. For example, we show, in Theorem 4.8, that if M admits an action by a semisimple Lie group such that some group element acts without fixed point, then $(f^*(x) \cdot P \cdot C)[M] = 0$.

We first review, in §4.1, techniques developed by [3], of which we note some immediate consequences. We then state and prove some "higher" vanishing theorems in §4.2.

4.1. **G-actions and the fundamental group.** In §4.1.1, we introduce [3]'s work relating the fundamental group of a G-manifold to the study of the G-action and we derive some direct consequences of it. In §4.1.2, we note an application.

4.1.1. *Relations among $H^*(M)$, $H^*(M/G)$, and $\pi_1(M)$.* Suppose G acts *effectively* on M. Let Γ be a (discrete) group and $s : M \to K(\Gamma, 1)$ a continuous map inducing a surjective homomorphism $s_* : \pi \to \Gamma$. Let $q : M \to M/G$ be the canonical quotient map and $o : G \to M$ the orbit of some basepoint of M. It is a standard fact that $o_*(\pi_1(G)) \subseteq \mathcal{Z}(\pi)$ (the center of π); from this and the surjectivity of s_*, it follows that $s_* o_*(\pi_1(G)) \subseteq \mathcal{Z}(\Gamma)$. Let $\Gamma_0 = \Gamma / s_* o_*(\pi_1(G))$ and let $\rho : \Gamma \to \Gamma_0$ be the quotient map.

Lemma 4.1 ([3])**.** *Let G, M, Γ, Γ_0, s, ρ, and q be as above. There exists a map*

$$\varphi : H^*(\Gamma_0; \mathbb{Q}) \longrightarrow H^*(M/G; \mathbb{Q})$$

making the following diagram commute:

$$\begin{array}{ccc} H^*(\Gamma_0; \mathbb{Q}) & \xrightarrow{\varphi} & H^*(M/G; \mathbb{Q}) \\ \downarrow \rho^* & & \downarrow q^* \\ H^*(\Gamma; \mathbb{Q}) & \xrightarrow{s^*} & H^*(M; \mathbb{Q}) \end{array}$$

φ is not necessarily induced by a map between spaces. Taking duals of the maps yields a similar commutative diagram of homology groups.

We note some consequences of this lemma.

If G is semisimple, or π has finite center, or $M^G \neq \varnothing$, then $s_* o_*(\pi_1(G))$ is finite, in which case

$$\rho^* : H^*(\Gamma_0; \mathbb{Q}) \longrightarrow H^*(\Gamma; \mathbb{Q})$$

is an isomorphism.

Another sufficient condition for the finiteness of $s_* o_*(\pi_1(G))$ is that the isotropy subgroup $G_a \stackrel{j}{\hookrightarrow} G$ be connected and $\pi_1(G) / j_* \pi_1(G_a)$ be finite. To see this, first note that the orbit map o factors as follows:

$$\begin{array}{ccccc} G & & \xrightarrow{o} & & M \\ & \searrow & & \nearrow & \\ & & G/G_a & & \end{array}$$

Note also that, for any connected subgroup $H \stackrel{j}{\hookrightarrow} G$, $\pi_1(G/H) \simeq \pi_1(G) / j_* \pi_1(H)$, which is easily seen via the long exact sequence of homotopy groups for the bundle $H \to G \to G/H$. Thus, under the stated condition, $o_*(\pi_1(G))$ is finite.

We summarize this discussion in

Corollary 4.2. *Let G, M, Γ, o, s, and q be as in Lemma* 4.1 ***except*** *that $\pi \stackrel{s_*}{\to} \Gamma$ need **not** be a surjection. Suppose $o_*(\pi_1(G))$ is finite, which is satisfied if **any** one of the following four conditions holds:*

(1) *G is semisimple;*

(2) $\mathcal{Z}(\pi)$ *is finite;*
(3) $M^G \neq \varnothing$*;*
(4) *there exists* $a \in M$ *such that its isotropy subgroup* $G_a \overset{j}{\hookrightarrow} G$ *is connected and* $\pi_1(G)/j_*\pi_1(G_a)$ *is finite.*

Then there exists a map

$$\varphi : H^*(\Gamma;\mathbb{Q}) \longrightarrow H^*(M/G;\mathbb{Q})$$

making the following diagram commute:

$$\begin{array}{ccc} H^*(\Gamma;\mathbb{Q}) & \xrightarrow{\varphi} & H^*(M/G;\mathbb{Q}) \\ s^* \searrow & & \swarrow q^* \\ & H^*(M;\mathbb{Q}) & \end{array}$$

Proof. When $\pi \overset{s_*}{\to} \Gamma$ is a surjection, this result follows from Lemma 4.1 once $H^*(\Gamma;\mathbb{Q})$ and $H^*(\Gamma_0;\mathbb{Q})$ are identified via the isomorphism ρ^*.

When $\pi \overset{s_*}{\to} \Gamma$ is not a surjection, s_* can be canonically factored into $\pi \overset{s'_*}{\to} \operatorname{Im} s_* \overset{i}{\hookrightarrow} \Gamma$, a surjection followed by an inclusion. As s'_* is a surjection, we may apply the above result to obtain a map φ' making the following diagram commute:

$$\begin{array}{ccc} H^*(\operatorname{Im} s'_*;\mathbb{Q}) & \xrightarrow{\varphi'} & H^*(M/G;\mathbb{Q}) \\ (s')^* \searrow & & \swarrow q^* \\ & H^*(M;\mathbb{Q}) & \end{array}$$

We can then forge another commutative diagram:

$$\begin{array}{ccc} H^*(\Gamma;\mathbb{Q}) & \xrightarrow{\varphi' i^*} & H^*(M/G;\mathbb{Q}) \\ i^* \downarrow & & \downarrow q^* \\ H^*(\operatorname{Im} s^*;\mathbb{Q}) & \xrightarrow{(s')^*} & H^*(M;\mathbb{Q}) \end{array}$$

Letting $\varphi = \varphi' i^*$, we have $q^*\varphi = (s')^* i^* = (is')^* = s^*$, as desired. □

We also mention that if $M \overset{f}{\to} K(\pi, 1)$ factors through M/G, then certainly a three-term diagram like the one in Corollary 4.2 (with $\Gamma = \pi$ and $s = f$) commutes. See [11] for a discussion of conditions that guarantee this.

Lemma 4.1 is stated under the assumption that the G-action on M is effective. However, we can allow the action $G \to \operatorname{Diff}(M)$ to have a nontrivial but finite kernel. Such an action is said to be almost effective.

Corollary 4.3. *If the* G*-action is almost effective, then the conclusion of Lemma* 4.1 *still holds.*

Proof. For convenience, let $N = \ker(G \to \operatorname{Diff}(M))$ and $G' = G/N$. Then $N \hookrightarrow G \overset{p}{\to} G'$ is a covering space of finite degree and hence $\pi_1(G')/p_*\pi_1(G)$ is finite. As we now have two actions on M, one being the almost effective G-action, and, the other the effective G'-action, we let $\Gamma_0(G)$ and $\Gamma_0(G')$ denote respectively $\Gamma/s_*o_*(\pi_1(G))$ and $\Gamma/s_*o'_*(\pi_1(G'))$, where $o' : G' \to M$ is the G'-orbit. Let $\rho' : \Gamma \to \Gamma_0(G')$ be the quotient map, and let h and h' abbreviate respectively s_*o_* and $s_*o'_*$. Note that $o = o'p$; hence, $h = h'p_*$.

Since G' acts effectively on M and $M/G = M/G'$, there is a map φ making the following diagram commute:

$$\begin{array}{ccc} H^*\left(\Gamma_0\left(G'\right);\mathbb{Q}\right) & \xrightarrow{\varphi} & H^*\left(M/G;\mathbb{Q}\right) \\ \downarrow \left(\rho'\right)^* & & \downarrow q^* \\ H^*\left(\Gamma;\mathbb{Q}\right) & \xrightarrow{s^*} & H^*\left(M;\mathbb{Q}\right) \end{array}$$

For our result, it suffices to prove that

$$H^*\left(\Gamma_0\left(G\right);\mathbb{Q}\right) \simeq H^*\left(\Gamma_0\left(G'\right);\mathbb{Q}\right).$$

We show this by showing that $\Gamma_0\left(G'\right)$ is a quotient of $\Gamma_0\left(G\right)$ by a finite subgroup.

We first note that

$$\Gamma_0\left(G'\right) \simeq \frac{\Gamma_0\left(G\right)}{h'\left(\pi_1\left(G'\right)\right)/h\left(\pi_1\left(G\right)\right)}.$$

This is simply a consequence of the definitions of the groups involved and the relevant isomorphism theorem in group theory. We next note that, since $p_*\pi_1\left(G\right)$ has finite index in $\pi_1\left(G'\right)$, $h'p_*\left(\pi_1\left(G\right)\right)$ (which equals $h\left(\pi_1\left(G\right)\right)$) also has finite index in $h'\left(\pi_1\left(G'\right)\right)$, and hence, $h'\left(\pi_1\left(G'\right)\right)/h\left(\pi_1\left(G\right)\right)$ is a finite (central) subgroup of $\Gamma_0\left(G\right)$, as desired. □

Remark 4.4. Suppose G is semisimple and $G \to \mathrm{Diff}\left(M\right)$ is a nontrivial G-action. Then, for some semisimple Lie group G', M admits an *effective* G'-action canonically induced by the given G-action: simply let $G' = G/N$, where $N = \ker\left(G \to \mathrm{Diff}\left(M\right)\right)$. As G has no nontrivial connected abelian normal subgroups, neither does G' and hence G' is semisimple.

Suppose $S \subset G$ is a circle subgroup acting nontrivially on M. $S \cap N$ must be finite; hence, S', the image of S in G', is also a circle subgroup, and $M^{S'} = M^S$.

We will need this observation in some later results.

4.1.2. *G-actions on aspherical manifolds and "the like".* Suppose M^m is such that $f_*\left[M^m\right] \neq 0 \in H_m\left(\pi;\mathbb{Q}\right)$. In light of the above discussion, we can gain insight into what Lie group actions M can admit. We first mention two earlier results.

[4] shows that, if M is aspherical, then the only Lie groups acting effectively on M are tori with dimensions bounded above by rank $\mathcal{Z}\left(\pi\right)$ ("$\mathcal{Z}$" signifying "the center of"), and such effective torus-actions must be fixed-point-free and have only finite isotropy subgroups. [12] shows that, if there exist $\omega_1, \cdots, \omega_m \in H^1\left(M\right)$ such that $\prod_j \omega_j[M^m] \neq 0$, then the only Lie groups acting effectively on M are tori.

Note that weaker than the hypotheses of both of the above results is the condition that $f_*\left[M\right] \neq 0$; thus, generalizing them is the next theorem, part (1) of which is observed by [3] and implies the other parts in a way similar to the development of [4].

Theorem 4.5. *Let M^m be a manifold satisfying $f_*\left[M\right] \neq 0 \in H_m\left(\pi;\mathbb{Q}\right)$.*

1. *If S^1 acts nontrivially on M and $o : S^1 \to M$ is the orbit of the basepoint in M, then $o_* : \pi_1\left(S^1\right) \to \pi$ is an injection.*
2. *If G is semisimple, then G cannot act nontrivially on M.*
3. *If G acts effectively on M, then G is a torus whose dimension is no greater than the rank of $\mathcal{Z}\left(\pi\right)$. Hence, if π has finite center, then all compact subgroups of* Diff (M) *are finite.*

(4) *If G acts nontrivially on M, then, for some $g \in G$, $M^g = \varnothing$ and hence $\chi(M)$ and all Pontrjagin numbers of M vanish. If, in addition, $H^1(M) = \{0\}$ then $(P \cdot C)[M] = 0$.*

(5) *If a torus T acts effectively on M, then all isotropy subgroups are finite.*

Proof. Suppose $S^1 \xrightarrow{\gamma} \mathrm{Diff}(M)$ is a nontrivial action. Then $\ker\gamma \subset S^1$ is finite, and Corollary 4.3 applies. If $o_*\left(\pi_1\left(S^1\right)\right)$ were finite, then Corollary 4.2 would give the following commutative diagram:

$$\begin{array}{ccc} H^m(\pi;\mathbb{Q}) & \xrightarrow{\varphi} & H^m\left(M/S^1;\mathbb{Q}\right) \\ f^* \searrow & & \swarrow q^* \\ & H^m(M^m;\mathbb{Q}) & \end{array}$$

As the action is nontrivial, $\dim M/S^1 = m-1$; hence, $H^m\left(M/S^1;\mathbb{Q}\right) = \{0\}$. But by hypothesis,

$$0 \neq f^* : H^m(\pi;\mathbb{Q}) \longrightarrow H^m(M;\mathbb{Q}).$$

This contradiction shows that $o_*\left(\pi_1\left(S^1\right)\right)$ must be infinite; as $\pi_1\left(S^1\right) \simeq \mathbb{Z}$, o_* is injective. This proves (1).

Notation. In the remainder of the proof, whenever G acts nontrivially on M, let $S \subset G$ be some circle subgroup of G that also acts nontrivially on M.

For (2), suppose the contrary. Let $o : S \to M$ be the S-orbit of the basepoint in M. Then o factors into $S \hookrightarrow G \xrightarrow{o'} M$, o' being the G-orbit map. As $\pi_1(G)$ is finite, $o_*(\pi_1(S))$ must also be finite, contradicting (1).

(3) follows from (2) and the structure of a compact connected Lie group G:

$$G \simeq \left(T \times \prod_{i=1}^{N} H_i\right)/\mathcal{C}$$

where T is a torus, H_i are simple Lie groups, and $\mathcal{C}$ is a finite central subgroup of the product. Suppose $G \xrightarrow{\gamma} \mathrm{Diff}(M)$ is an effective action. If G is not toral, then, each H_j would act nontrivially on M via

$$H_j \hookrightarrow \left(T \times \prod H_i\right) \longrightarrow G \xrightarrow{\gamma} \mathrm{Diff}(M),$$

contradicting (2). Hence, $G = T^n$. To see that $n \leq \operatorname{rank} \mathcal{Z}(\pi)$, we let $o : T^n \to M$ be the T^n-orbit and show that $o_* : \pi_1(T^n) \to \pi$ is injective, which will yield the desired inequality. We simply note that a nontrivial element in $\ker o_*$ can be realized as a nontrivial Lie group homomorphism $S^1 \xrightarrow{\alpha} T^n$, which gives rise to a nontrivial S^1-action on M via $S^1 \xrightarrow{\alpha} T^n \xrightarrow{\gamma} \mathrm{Diff}(M)$. $o\alpha$ is then the S^1-orbit. But $(o\alpha)_*[\mathrm{Id}_{S^1}] = o_*[\alpha] = 0$ by the choice of α. Hence, $\pi_1\left(S^1\right) \xrightarrow{(o\alpha)_*} \pi$ is then trivial, contradicting (1).

For (4), let $g \in S$ be a topological generator of S. Then, $M^S = M^g$. If $M^S \neq \varnothing$, by picking a fixed point as the basepoint, the orbit map $S \to M$ is constant, contradicting (1). The vanishing of the Euler characteristic and Pontrjagin numbers follows readily. With the condition that $H^1(M) = \{0\}$, Theorem 3.11 implies the vanishing of $(P \cdot C)[M]$.

(5) also follows from (1). For $a \in M$, let T_a be the isotropy subgroup of a. If T_a were infinite, then its identity component would contain a circle group acting effectively on M fixing a, contradicting (1). $\square$

A similar argument would show

Proposition 4.6. *Suppose M is a manifold with an effective G-action having only finite isotropy groups. Let*

$$d = \max\left\{j | 0 \neq f^* : H^j(\pi;\mathbb{Q}) \longrightarrow H^j(M;\mathbb{Q})\right\}.$$

If either G is semisimple or $\mathcal{Z}(\pi)$ is finite, then $\dim G \leq \dim M - d$.

Proof. When G acts on M with finite isotropy groups, M/G is a rational homology manifold with $\dim M/G = \dim M - \dim G$. As either G is semisimple or $\mathcal{Z}(\pi)$ is finite, Corollary 4.2 applies. If $\dim G > \dim M - d$, then $H^d(M/G;\mathbb{Q}) = \{0\}$ and hence $0 = f^* : H^d(\pi;\mathbb{Q}) \to H^d(M;\mathbb{Q})$, which is contrary to the hypothesis. $\square$

4.2. **Higher vanishing theorems.** We now turn to the main results of the present section.

First we introduce a lemma implicit in [3].

A G-submanifold K in a G-manifold M is said to be transverse if K meets M^H transversally for every closed subgroup $H \subset G$.

Lemma 4.7 ([3])**.** *Let M be a G-manifold. For each nonzero*

$$y \in \operatorname{Im}\left(H^l(M/G) \to H^l(M/G;\mathbb{Q})\right),$$

there exists an l-codimensional framed transverse G-submanifold $\iota : K_y \hookrightarrow M$ such that $q^*(y) \frown [M] = \iota_*[K_y]$ where $q : M \to M/G$ is the quotient map.

We are now ready to state the main results.

Theorem 4.8. *Suppose M^m admits a nontrivial action by a semisimple Lie group G such that there exists a circle subgroup $S \subset G$ with $M^S = M^G$. Let $x \in H^{m-2n}(\pi;\mathbb{Q})$ be a nonzero class such that*

$$\varphi(x) \in \operatorname{Im}\left(H^{m-2n}(M/G) \to H^{m-2n}(M/G;\mathbb{Q})\right)$$

where

$$\varphi : H^{m-2n}(\pi;\mathbb{Q}) \to H^{m-2n}(M/G;\mathbb{Q})$$

is as in Corollary 4.2. Then, $(f^*(x) \cdot P \cdot C)[M]$ can be determined in terms of $K^S_{\varphi(x)} \overset{\iota}{\hookrightarrow} K_{\varphi(x)}$ and the S-action on the normal bundle of $K^S_{\varphi(x)} \hookrightarrow K_{\varphi(x)}$, where $K_{\varphi(x)}$ is as in Lemma 4.7.

Proof. By Remark 4.4, we may assume that the action is effective; by Remark 3.2 and Corollary 4.3, we may assume that G is simply connected.

If $n = 0$, i.e., if $x \in H^m(\pi;\mathbb{Q})$, then, by Theorem 4.5(2), $f^*(x)[M] = x(f_*[M]) = 0$. So assume $n > 0$.

Since $K_{\varphi(x)} \hookrightarrow M$ is a submanifold with trivial normal bundle, we identify $\iota^*(p_i)$ with $p_i\left(K_{\varphi(x)}\right)$. With this in mind and using Corollary 4.2 and Lemma 4.7, we compute:

$$\begin{aligned}(4.1)\qquad &\left(f^*(x)\cdot P(p_1,\cdots,p_{[m/4]})\cdot C\right)[M]\\ &=\left(q^*\varphi\left(x\right)\cdot P\cdot C\right)[M]\\ &=\left(P\cdot C\right)\left(q^*\varphi\left(x\right)\frown[M]\right)\\ &=\left(P\cdot C\right)\iota_*\left[K_{\varphi(x)}\right]\\ &=\iota^*\left(P\cdot C\right)\left[K_{\varphi(x)}\right]\\ &=\left(P\left(\iota^*\left(p_1\right),\cdots,\iota^*\left(p_{[m/4]}\right)\right)\cdot\iota^*\left(C\right)\right)\left[K_{\varphi(x)}\right]\\ &=\left(P\left(p_1\left(K_{\varphi(x)}\right),\cdots,p_{[m/4]}\left(K_{\varphi(x)}\right)\right)\cdot\iota^*\left(C\right)\right)\left[K_{\varphi(x)}\right].\end{aligned}$$

Since $K_{\varphi(x)}$ is a G-submanifold of M and $M^S=M^G$, we have $K^S_{\varphi(x)}=K^G_{\varphi(x)}$. Applying Theorem 3.6 to $K_{\varphi(x)}$ yields the desired result. □

Corollary 4.9. *If M^m admits an action by a semisimple Lie group G such that, for some $g\in G$, $M^g=\varnothing$, then $\left(f^*\left(x\right)\cdot P\cdot C\right)[M]=0$.*

Proof. Just as in the proof of Corollary 3.7, we may assume, without any loss, that g generates a circle subgroup $S\subset G$. Hence, $M^S=\varnothing=M^G$. Applying Theorem 4.8 yields the desired conclusion. □

In the following, for an S^1-manifold M, let $o:S^1\to M$ be the orbit of the basepoint and $o_*:\pi_1\left(S^1\right)\to\pi_1\left(M\right)$ the induced map.

Theorem 4.10. *Let M^m be a fixed-point-free S^1-manifold and $x\in H^*(\pi;\mathbb{Q})$.*

1. *Suppose $\operatorname{Im}o_*$ is finite. Then $\left(f^*(x)\cdot P\right)[M]=0$; if, in addition, $H^1\left(M\right)=\{0\}$, then $\left(f^*(x)\cdot P\cdot C\right)[M]=0$.*
2. *Suppose, for some semisimple Lie group G, the S^1-action extends to a G-action, then $\left(f^*(x)\cdot P\cdot C\right)[M]=0$.*

Proof. As noted earlier, under the hypotheses of both parts of the theorem, for $x\in H^m(\pi;\mathbb{Q})$, $f^*(x)[M]=x\left(f_*\left[M\right]\right)=0$ by Theorem 4.5(2). So assume $x\in H^{m-2n}(\pi;\mathbb{Q})$ with $n>0$; also assume without loss of generality that x satisfies the condition in Theorem 4.8.

A nontrivial action $S^1\to\operatorname{Diff}\left(M\right)$ always has a finite kernel, making Corollary 4.3 applicable; finiteness of $\operatorname{Im}o_*$ makes Corollary 4.2 applicable. A computation similar to that in (4.1) yields

$$\begin{aligned}&\left\{f^*(x)\cdot P\left(p_1,\cdots,p_{[m/4]}\right)\right\}[M]\\ &=\left\{q^*\varphi(x)\cdot P\left(p_1,\cdots,p_{[m/4]}\right)\right\}[M]\\ &=P\left(p_1\left(K_{\varphi(x)}\right),\cdots,p_{[m/4]}\left(K_{\varphi(x)}\right)\right)\left[K_{\varphi(x)}\right]\end{aligned}$$

where $\varphi:H^{m-2n}(\pi;\mathbb{Q})\to H^{m-2n}(M/S^1;\mathbb{Q})$ is as in Corollary 4.2 and $K_{\varphi(x)}\overset{\iota}{\hookrightarrow}M$ is as in Lemma 4.7. As $K^{S^1}_{\varphi(x)}=K_{\varphi(x)}\cap M^{S^1}$, $K^{S^1}_{\varphi(x)}=\varnothing$; hence Remark 3.10 shows

$$P\left(p_1\left(K_{\varphi(x)}\right),\cdots,p_{[m/4]}\left(K_{\varphi(x)}\right)\right)\left[K_{\varphi(x)}\right]=0,$$

proving the first statement of (1).

For the second statement of (1), we note, as in (4.1),

$$\begin{aligned}&\left(f^*(x)\cdot P(p_1,\cdots,p_{[m/4]})\cdot C\right)[M]\\ &=\left(P\left(p_1\left(K_{\varphi(x)}\right),\cdots,p_{[m/4]}\left(K_{\varphi(x)}\right)\right)\cdot\iota^*\left(C\right)\right)\left[K_{\varphi(x)}\right].\end{aligned}$$

The condition $H^1(M) = \{0\}$ makes Corollary 2.2 applicable. Hence, for $C = \prod_j c_j$ where $c_j \in \operatorname{Im}\left(H^2(M) \to H^2(M;\mathbb{Q})\right)$, the S^1-action on M admits a lifting to an action on each complex line bundle $\mathcal{L}(c_j)$, from which the bundle $\iota^*\mathcal{L}(c_j)$ inherits an S^1-equivariant structure. Applying the proof of Theorem 3.6 to the S^1-manifold $K_{\varphi(x)}$ yields the result.

(2) is immediate from Corollary 4.9. □

When $\operatorname{Im} o_*$ is infinite, we may resort to the next theorem, which can be shown using Lemma 4.1 and following the proof of the last theorem.

Theorem 4.11. *Suppose M is a fixed-point-free S^1-manifold. Let $\pi' = \pi/\operatorname{Im} o_*$, let $\rho : \pi \to \pi'$ be the quotient map, and let $y \in H^*(\pi';\mathbb{Q})$. Then, $(f^*\rho^*(y) \cdot P)[M] = 0$; if, in addition, $H^1(M) = \{0\}$, then, $(f^*\rho^*(y) \cdot P \cdot C)[M] = 0$.*

5. Vanishing theorems on spin G-manifolds

We study obstructions to certain G-actions on spin manifolds.

If M is a spin manifold admitting a nontrivial S^1-action, then [1] shows that its $\hat{\mathbf{A}}$-genus vanishes; [3] proves the vanishing of *certain* "higher $\hat{\mathbf{A}}$-genera". If M is a spin manifold admitting a spin-structure-preserving action by a semisimple Lie group G, then, as a consequence of [3]'s result, *all* "higher $\hat{\mathbf{A}}$-genera" vanish; similar vanishing theorems hold for finer invariants with values in K-theory, such as $\hat{\mathcal{A}}$ and G-$\hat{\mathcal{A}}$; see [9] and [11] for details.

In this section, we show some similar results.

As before, let $\pi = \pi_1(M)$, let $f : M \to K(\pi, 1)$ classify the universal cover of M, let $x \in H^*(\pi;\mathbb{Q})$, and let C be an element of the subalgebra of $H^*(M;\mathbb{Q})$ generated by $H^2(M;\mathbb{Q})$.

Theorem 5.1. *Let G be a semisimple Lie group acting on a spin manifold M^{2n}. Suppose there exists a circle subgroup $S \subset G$ acting nontrivially on M with every component of M^S containing a point of M^G. Then $\left(\hat{\mathbf{A}}(M) \cdot C\right)[M] = 0$.*

Proof. By Remark 3.2, we assume, without loss of generality, that G is simply connected, and G acts isometrically on M. As G is now assumed to be simply connected, the G-action necessarily preserves the spin structure on M.

In the following, we adopt the same notation as in the proof of Theorem 3.1.

Let $c \in \operatorname{Im}\left(H^2(M) \to H^2(M;\mathbb{Q})\right)$. By the G-spin Theorem 2.8, for z a topological generator of S,

$$\begin{aligned}
&\operatorname{Spin}_G(M;\mathcal{L}(c))(z)\\
&= \operatorname{Spin}_S(M;\mathcal{L}(c))(z)\\
&= \sum_\nu \left\{ \sigma_\nu z^{l_\nu} e^{i_\nu^*(c)} \hat{\mathbf{A}}\left(M_\nu^S\right) \prod_{k\in\mathbb{N}} \prod_{j=1}^{d_\nu(k)} \frac{z^{\frac{k}{2}} e^{-\frac{1}{2}x_{\nu,j}(k)}}{1 - z^k e^{-x_{\nu,j}(k)}} \right\} \left[M_\nu^S\right],
\end{aligned}$$

$\operatorname{Spin}_S(M;\mathcal{L}(c))(z) \in R(S)$; it is thus a Laurent polynomial $w(z)$. As explained in the proof of Theorem 3.1, $l_\nu = 0$. Therefore, by the above formula, $w(0) = 0 =$

$w(\infty)$. Hence, $w(z) = 0$ for all $z \in \mathbb{C}$. By Theorem 2.9,

$$\begin{aligned} 0 &= w(1) \\ &= 2^n \left(\mathrm{ch}(\mathcal{L}(c))\, \hat{\mathbf{A}}(M) \right)[M] \\ &= 2^n \left(e^c \cdot \hat{\mathbf{A}}(M) \right)[M]. \end{aligned}$$

It follows then, just as in the proof of Theorem 3.1, that $\left(\hat{\mathbf{A}}(M) \cdot c^d\right)[M] = 0$. Using Lemma 3.4 yields the desired result. □

Using techniques of §4, this result can be modified to produce an obstruction from the fundamental group.

Theorem 5.2. *Let G be a semisimple Lie group acting on a spin manifold M^m. Suppose there exists a circle subgroup $S \subset G$ acting nontrivially on M with $M^S = M^G$. Then, $\left(f^*(x) \cdot \hat{\mathbf{A}}(M) \cdot C\right)[M] = 0$.*

Proof. Let $x \in H^{m-2n}(\pi; \mathbb{Q})$. If $n = 0$, then, by Theorem 4.5(2), $f^*(x)[M] = x(f_*[M]) = 0$. So assume $n > 0$. Also assume, without loss of generality, that x satisfies the condition in Theorem 4.8. As in (4.1),

$$\begin{aligned} &\left(f^*(x) \cdot \hat{\mathbf{A}}(M) \cdot C\right)[M] \\ &= \left(q^*\varphi(x) \cdot \hat{\mathbf{A}}(M) \cdot C\right)[M] \\ &= \left(\hat{\mathbf{A}}(M) \cdot C\right)(q^*\varphi(x) \frown [M]) \\ &= \left(\hat{\mathbf{A}}(M) \cdot C\right)\iota_*\left[K_{\varphi(x)}\right] \\ &= \iota^*\left(\hat{\mathbf{A}}(M) \cdot C\right)\left[K_{\varphi(x)}\right] \\ &= \left(\hat{\mathbf{A}}(K_{\varphi(x)}) \cdot \iota^*(C)\right)\left[K_{\varphi(x)}\right]. \end{aligned}$$

Since $K_{\varphi(x)}$ has trivial normal bundle in a spin manifold M, $K_{\varphi(x)}$ is also a spin manifold; as $K_{\varphi(x)}$ is a transverse G-submanifold, the S-action on it is nontrivial. Hence, Theorem 5.1, applied to $K_{\varphi(x)}$, yields the result. □

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Department of Mathematics, Indiana University, Bloomington, Indiana 47405
E-mail address: pding@indiana.edu